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PLANAR λ CONNECTED CONTINUA

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ABSTRACT. Certain fixed point and mapping properties of arcwise connected plane continua have recently been established for planar λ connected continua. We show that properties involving union, product, and intersection operations are also shared by arcwise and λ connected plane continua.

By a continuum we mean a compact connected nondegenerate metric space. A continuum M is λ connected if any two of its points can be joined by a hereditarily decomposable continuum in M. It is known that every λ connected nonseparating plane continuum has the fixed point property [**1**]. In [**3**] it is proved that every planar continuous image of a λ connected continuum is λ connected. Here we show that λ connected plane continua share other properties with arcwise connected continua. In fact, we find that by making the transition from arcwise to λ connected plane continua certain theorems are improved. For example, when a plane continuum M is the union of finitely many arcwise connected continua, it follows that M is arcwise connected. To see that this theorem does not extend to countable unions we need only consider the topologist's sine curve. However, every plane continuum that is the union of countably many λ connected continua is λ connected.

We show that the topological product of countably many plane continua is λ connected if and only if each factor is λ connected. It follows from Theorem 6 of this paper that if a λ connected continuum M and a disk D lie in a plane, then each nondegenerate component of $M\cap D$ is λ connected.

DEFINITIONS. Let M be a plane continuum. A subcontinuum L of M is said to be a *link* in M if L is either the boundary of a complementary domain of M or the limit of a convergent sequence of complementary domains of M. An indecomposable subcontinuum I of M is said to be *terminal* in M if there exists a composant C of I such that each subcontinuum of M that meets both C and $M-I$ contains I.

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THEOREM 1. *Suppose that the union of finitely many links in a plane continuum M contains an indecomposable continuum I. Then I is contained in a link in M.*

PROOF. According to the Baire category theorem, there is a link L in M that contains an open subset of I. Since L is a subcontinuum of M, it follows that L contains I [3, Theorem 1].

THEOREM 2. *A plane continuum M is λ connected if and only if each link in M is hereditarily decomposable.*

PROOF. Suppose there exists a link L in M that contains an indecomposable continuum I. Then I is terminal in M ([2, Theorem 2], [3, Theorem 1]). Hence there exist points x and y of M such that every subcontinuum of M that contains $\{x, y\}$ also contains I. Therefore M is not λ connected.

Assume each link in M is hereditarily decomposable. It follows from Theorem 1 that no finite collection of links in M contains an indecomposable continuum in its union. Consequently M is λ connected [3, Theorem 3].

DEFINITION. Let y be a point of a continuum M. If there exist distinct points x and z in $M-\{y\}$ such that each subcontinuum of M that contains x and z also contains y, then y is said to be a *cut point* of M.

It follows from a theorem due to F. Burton Jones [4] that every nonseparating plane continuum that does not have a cut point is arcwise connected. Jones points out in [4] that a planar continuum that has no cut point but separates the plane may fail to be arcwise connected.

THEOREM 3. *If a plane continuum M does not have a cut point, then M is λ connected.*

PROOF. Assume that M is not λ connected. According to Theorem 2, some link in M contains an indecomposable continuum I. It follows from [2, Theorem 2] and [3, Theorem 1] that I is terminal in M. Since this implies that some point of I is a cut point of M, we have a contradiction. Hence M is λ connected.

THEOREM 4. *If a plane continuum M is the union of countably many λ connected continua, then M is λ connected.*

PROOF. Suppose that M is not λ connected. Let K be a countable collection of λ connected continua whose union is M. There exists a link in M that contains an indecomposable continuum I (Theorem 2). Since K is countable, there is an element H of K that contains an open subset of I. It follows that I is a subcontinuum of H [3, Theorem 1]. Since I is in a link in M and H is a subcontinuum of M that contains I, a link in

H contains I. This contradicts the assumption that H is λ connected (Theorem 2). Therefore M is λ connected.

EXAMPLE 1. Theorem 4 does not hold for λ connected continua in Euclidean 3-space. In fact, the union of two intersecting λ connected continua in E^3 may fail to be λ connected. Consider the 3-space continua E and J in Figure 1. E is the closure of a ray that starts at a point p and limits on a "V shaped" arc A and a subset of Knaster's indecomposable continuum Y [**5**, Example 1, p. 205]. J is the closure of a ray missing E from a point q that limits on A and $Y-E$. Each interval indicated by a dotted line in the figure is considered to be a point. Both E and J are hereditarily decomposable. Each subcontinuum of $E\cup J$ that joins p and q (there is only one, $E\cup J$ itself) contains Y. Hence $E\cup J$ is not λ connected.

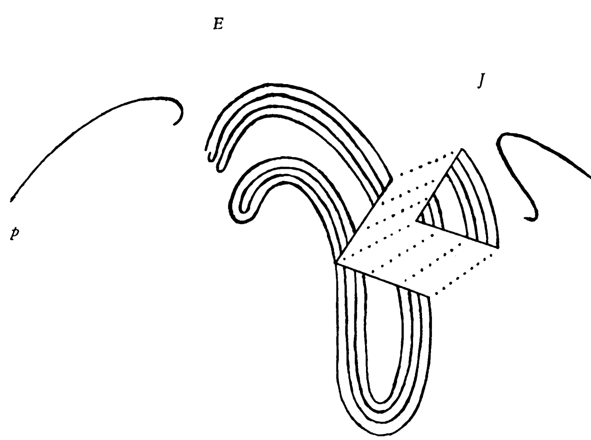

FIGURE 1

THEOREM 5. *Suppose that* $\{M_i\}$ *is a countable collection of plane continua. Then the topological product* $\times M_i$ *is* λ *connected if and only if each element of* $\{M_i\}$ *is* λ *connected.*

PROOF. If $\times M_i$ is λ connected, then each factor, being a planar continuous image of $\times M_i$, is λ connected [**3**, Theorem 5].

Now suppose that $\{M_i\}$ is an infinite collection of λ connected continua. Assume $(x_1, x_2, \cdots)$ and $(y_1, y_2, \cdots)$ are distinct points of $\times M_i$. For each i, there exists a hereditarily decomposable subcontinuum H_i of M_i that contains the set $\{x_i, y_i\}$. For each positive integer n, let E_n be the subcontinuum of $\times M_i$ consisting of all points $(z_1, z_2, \cdots)$ such that $z_i=y_i$ if $i<n$, z_i belongs to H_i if $i=n$, and $z_i=x_i$ if $i>n$. If i and j are distinct positive integers, then $E_i\cap E_j$ consists of at most one point. Furthermore, $E_i\cap E_j=\varnothing$ if and only if $|i-j|>1$. Since the sequence

$\{E_n\}$ converges to $(y_1, y_2, \cdots)$ in $\times M_i$, the closure of $\bigcup E_n$ is a hereditarily decomposable subcontinuum of $\times M_i$ that contains $(x_1, x_2, \cdots)$ and $(y_1, y_2, \cdots)$.

Suppose that $\{M_i\}$ consists of only finitely many λ connected continua. It follows from an argument similar to the preceding that any two points of $\times M_i$ can be joined by a hereditarily decomposable continuum in $\times M_i$. Hence $\times M_i$ is λ connected.

EXAMPLE 2. Note that the last arguments in the proof of Theorem 5 hold when the factors are not assumed to be planar. Hence the product of any countable collection of λ connected continua is λ connected. However, a λ connected product may have a factor that is not λ connected. The product resulting from crossing the 3-space continuum $E \cup J$ (defined in Example 1) with an arc is λ connected.

If D is a disk in a plane S and M is a nonseparating arcwise connected continuum in S, then each component of $M \cap D$ is arcwise connected. Although this theorem can not be extended to all arcwise connected plane continua [7, Example 4, p. 230], its analogue does hold for every planar λ connected continuum.

THEOREM 6. *Suppose that a plane S contains a λ connected continuum M and a disk D. Then each nondegenerate component of $M \cap D$ is λ connected.*

PROOF. Assume there exists a nondegenerate component X of $M \cap D$ that is not λ connected. According to Theorem 2, there is a link L in X that contains an indecomposable continuum I. The union of the composants of I that meet the boundary of D is a first category subset of I [**6**]. Furthermore, each subcontinuum of X that contains a nonempty open subset of I contains I [**3**, Theorem 1]. Define K to be the collection consisting of all composants Z of I such that a continuum in X meets both Z and $X-I$ and does not contain I. The union of the elements of K is a first category subset of I [**2**, Theorem 2 (Proof)]. Hence there exists a composant C of I such that the boundary of D does not meet C and each subcontinuum of X that meets both C and $X-I$ contains I.

Suppose there is a sequence $\{R_i\}$ of complementary domains of X that converges to L. Since only finitely many elements of $\{R_i\}$ meet $S-D$, there exists a subsequence of $\{R_i\}$, each element of which is a complementary domain of M. Therefore L is a link in M. Since L contains I and M is λ connected, this is a contradiction (Theorem 2). Hence there is a complementary domain R of X whose boundary is L. Note that R is not a complementary domain of M (Theorem 2).

Let G be a circular region in D centered on a point p of C. Suppose that no complementary domain of M meets both G and $S-D$. It follows that X is the only component of $M \cap D$ that intersects G [7, Theorem 14,

p. 171]. Hence the boundary of each complementary domain of M that meets G is a subcontinuum of X. Therefore each complementary domain of X that meets G is a complementary domain of M. Since R is a complementary domain of X that meets G, this is a contradiction. Evidently a complementary domain of M meets both G and $S-D$.

For each positive integer i, let G_i be a circular region in D of radius less than i^{-1} centered on p. The preceding paragraph indicates that, for each i, there exists a complementary domain U_i of M that meets both G_i and $S-D$. Using a simple closed curve approximation of the boundary of U_i from the inside, we define, for each i, an arc A_i in $U_i \cap D$ such that G_i and the boundary of D meet A_i and the distance from any point of A_i to M is less than i^{-1}. There is a subsequence of $\{A_i\}$ that converges to a continuum W in $M \cap D$. The continuum W is in X, contains p, and meets the boundary of D. Therefore W contains I. Since the elements of $\{A_i\}$ lie in the complement of M, it follows that W is a subcontinuum of some link in M. This contradicts the assumption that M is λ connected (Theorem 2). Hence X is λ connected.

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