ON THE LOWER BOUND OF THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION WITH INFINITE VARIANCE. III

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ABSTRACT. Let N_n be the number of real roots of a random algebraic equation $\sum_0^n a_v \xi_v x^v = 0$, where ξ_v 's are independent random variables with common characteristic function $\exp(-C|t|^\alpha)$, C being a positive constant, $\alpha \geq 1$ and $a_0, a_1, \dots, a_n$ are nonzero real numbers. Let $\kappa_n = \max_{0 \leq v \leq n} |a_v|$, $t_n = \min_{0 \leq v \leq n} |a_v|$. If $\Lambda_n = \mu \log n / \log((\kappa_n/t_n) \log n)$, then

$$(i) \quad \Pr\{N_n/\Lambda_n \geq \mu\} < \frac{\mu}{\log((\kappa_n/t_n) \log n)(\log n)^{\alpha-1}}$$

if $1 \leq \alpha < 2$ and $< \mu \log((\kappa_n/t_n) \log n) / \log n$ if $\alpha > 2$, provided $(\kappa_n/t_n) = o(\log n)$.

$$(ii) \quad \Pr\left\{\sup_{n > n_0} \frac{N_n}{\Lambda_n} \geq \mu\right\} < \mu' / \left\{\log\left(\frac{\log n_0}{\log((\kappa_{n_0}/t_{n_0}) \log n)}\right)\right\}^{\alpha-1}$$

if $\alpha > 1$, provided $\log(\kappa_n/t_n) = o(\log n)$.

It may be remarked that the coefficients $a_v \xi_v$'s in the above equation are not identically distributed.

1. Introduction. In Samal and Mishra ([3] and [4]) we have considered the lower bound of the number of real roots of the algebraic equation

$$(1) \quad f(x) = \xi_0 + \xi_1 x + \dots + \xi_n x^n = 0$$

where ξ_v 's are identically distributed independent random variables with a common characteristic function $\exp(-C|t|^\alpha)$, C is a positive constant and $\alpha \geq 1$. In fact we have proved that

$$(i) \quad N_n \geq \mu \log n / \log \log n.$$

The measure of the exceptional set is $\mu / (\log \log n)(\log n)^{\alpha-1}$ if $1 \leq \alpha < 2$ and $\mu \log \log n / \log n$ if $\alpha \geq 2$,

$$(ii) \quad \text{for } n > n_0, \quad N_n > \mu \log n / \log \log n$$

outside a set of measure at most $\mu' / \{\log(\log n_0 / \log \log n_0)\}^{\alpha-1}$.

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Our present work is to indicate how the same technique with a little modification can be used to extend our results to the case of algebraic equations with nonidentically distributed coefficients. We consider here the algebraic equation

$$(2) \quad f(x) = a_0 \xi_0 + a_1 \xi_1 x + a_2 \xi_2 x^2 + \cdots + a_n \xi_n x^n = 0$$

in the place of (1), where $a_0, a_1, a_2, \dots, a_n$ are nonzero real numbers with some restrictions. The coefficients $a_0 \xi_0, a_1 \xi_1, \dots, a_n \xi_n$ are not identically distributed, although $\xi_0, \xi_1, \dots, \xi_n$ have the same distribution.

Equations of the type (2) have been considered by Dunnage [1]. But his results do not cover our case in which the variance is infinite.

2. THEOREM 1. *Let $f(x) = a_0 \xi_0 + a_1 \xi_1 x + a_2 \xi_2 x^2 + \cdots + a_n \xi_n x^n$ be a polynomial where ξ_v 's are independent random variables identically distributed with a common characteristic function $\exp(-C|t|^\alpha)$, C is a positive constant and $\alpha \geq 1$. $a_0, a_1, a_2, \dots, a_n$ are nonzero real numbers such that $(\kappa_n/t_n) = o(\log n)$ where $\kappa_n = \max_{0 \leq v \leq n} |a_v|$, $t_n = \min_{0 \leq v \leq n} |a_v|$.*

Then for $n \geq n_0$ the number of real roots of $f(x) = 0$ is at least $(\mu \log n)/\log((\kappa_n/t_n) \log n)$ outside a set of measure at most

$$\mu / \{ \log((\kappa_n/t_n) \log n) \} (\log n)^{\alpha-1} \quad \text{if } 1 \leq \alpha < 2$$

and

$$\mu \log((\kappa_n/t_n) \log n) / \log n \quad \text{if } \alpha \geq 2.$$

PROOF. We indicate below only those steps of Samal and Mishra [3] which need modification. The steps not mentioned here remain unchanged.

Let M be the integer defined by

$$M = [2^{1+\alpha} A e (\kappa_n/t_n)^{\alpha} \lambda^\alpha / B] + 1.$$

We take

$$U_m = \sum_{M^{2m-1}+1}^{M^{2m+1}} a_v \xi_v x_m^v, \quad R_m = \left(\sum_0^{M^{2m-1}} + \sum_{M^{2m+1}+1}^n \right) a_v \xi_v x_m^v$$

and

$$V_m = \left(\sum_{M^{2m-1}+1}^{M^{2m+1}} |a_v|^\alpha x_m^{\alpha v} \right)^{1/\alpha}.$$

The characteristic function of U_m is

$$\exp \left\{ -C |t|^\alpha \sum_{M^{2m-1}+1}^{M^{2m+1}} |a_v|^\alpha x_m^{\alpha v} \right\}.$$

LEMMA 1.2.

$$\left| \sum_{M^{2m-1}+1}^n a_v \xi_v x_m^v \right| < \frac{1}{2} V_m$$

except for a set of measure at most

$$\left(\frac{\kappa_n}{t_n}\right)^\alpha \cdot \frac{2^{1+2\alpha} \cdot C}{1 + \alpha} \cdot \frac{Ae}{B} e^{-M}.$$

PROOF. The characteristic function of $\sum_{M^{2m+1}+1}^n a_v \xi_v x_m^v$ is

$$\exp\left\{-C |t|^\alpha \sum_{M^{2m+1}+1}^n |a_v|^\alpha x_m^{\alpha v}\right\}.$$

$$\Pr\left\{\left|\sum_{M^{2m+1}+1}^n a_v \xi_v x_m^v\right| \geq \frac{1}{2} V_m\right\} \leq \left(C \cdot 2^{1+2\alpha} \sum_{M^{2m+1}+1}^n |a_v|^\alpha x_m^{\alpha v}\right) / (1 + \alpha) V_m^\alpha.$$

But

$$\sum_{M^{2m+1}+1}^n |a_v|^\alpha x_m^{\alpha v} \leq \kappa_n^\alpha M^{2m} e^{-M}$$

and $V_m^\alpha \geq t_n^\alpha M^{2m} B(e^{-1}/A)$. Therefore

$$P_1 \leq \frac{2^{1+2\alpha} \cdot C}{1 + \alpha} \cdot \frac{Ae}{B} \cdot \left(\frac{\kappa_n}{t_n}\right)^\alpha \cdot e^{-M}.$$

LEMMA 1.3.

$$\left|M \sum_0^{2m-1} a_v \xi_v x_m^v\right| < \lambda \left(M \sum_0^{2m-1} |a_v|^\alpha x_m^{\alpha v}\right)^{1/\alpha}$$

except for a set of measure at most $(2^{1+\alpha} \cdot C)/(1 + \alpha)\lambda^\alpha$.

The characteristic function of $\sum_0^{M^{2m-1}} a_v \xi_v x_m^v$ is

$$\exp\left\{-C |t|^\alpha \sum_0^{M^{2m-1}} |a_v|^\alpha x_m^{\alpha v}\right\}.$$

Hence

$$P_2 = \Pr\left\{\left|M \sum_0^{2m-1} a_v \xi_v x_m^v\right| \geq \lambda \left(M \sum_0^{2m-1} |a_v|^\alpha x_m^{\alpha v}\right)^{1/\alpha}\right\} \leq \frac{2^{1+\alpha} \cdot C}{(1 + \alpha)\lambda^\alpha}.$$

Therefore $|R_m| < \frac{1}{2} V_m + \lambda \left(\sum_0^{M^{2m-1}} |a_v|^\alpha x_m^{\alpha v}\right)^{1/\alpha}$, except for a set of measure at most $\mu_1(\kappa_n/t_n)^\alpha e^{-M} + \mu_2/\lambda^\alpha$. But

$$\lambda \left(M \sum_0^{2m-1} |a_v|^\alpha x_m^{\alpha v}\right)^{1/\alpha} \leq \lambda \frac{\kappa_n}{t_n} \left(\frac{2Ae}{BM}\right)^{1/\alpha} V_m < \frac{1}{2} V_m.$$

The measure of the exceptional set is at most

$$\begin{aligned} & \mu/k + \mu_1 \cdot \frac{1}{2}(k+1)(\kappa_n/t_n)^\alpha e^{-M} + \mu_2 \cdot \frac{1}{2}(k+1)\lambda^{-\alpha} \\ & < \mu/k + \mu_1 k(\kappa_n/t_n)^\alpha e^{-M} + \mu_2 k\lambda^{-\alpha}. \end{aligned}$$

This corresponds to (11) of Samal and Mishra [2]. From the definition of M and k it follows that

$$\mu_3\{(\kappa_n/t_n)\log n\}^\alpha \leq M \leq \mu_4\{(\kappa_n/t_n)\log n\}^\alpha$$

and

$$\frac{\mu_5 \log n}{\log\{(\kappa_n/t_n)\log n\}} \leq k \leq \frac{\mu_6 \log n}{\log\{(\kappa_n/t_n)\log n\}}.$$

Since $(\kappa_n/t_n) = o(\log n)$ we have $\log(\kappa_n/t_n)/\log n \rightarrow 0$. So $k \rightarrow \infty$ as $n \rightarrow \infty$. Again $\kappa_n/t_n > 1$, so $(\kappa_n/t_n)\log n \rightarrow \infty$ as $n \rightarrow \infty$. Therefore it follows that $M \rightarrow \infty$ as $n \rightarrow \infty$.

Now

$$\begin{aligned} \{k(\kappa_n/t_n)^\alpha e^{-M}\}/(k\lambda^{-\alpha}) &= (\kappa_n/t_n)^\alpha \lambda^\alpha e^{-M} \\ &\leq (\kappa_n/t_n)^\alpha (\log n)^\alpha \exp\{-\mu_3(\kappa_n/t_n \log n)^\alpha\}. \end{aligned}$$

The right-hand side tends to zero as $n \rightarrow \infty$. Hence $k(\kappa_n/t_n)^\alpha e^{-M} < k\lambda^{-\alpha}$ when n is large. Again

$$\begin{aligned} (k\lambda^{-\alpha})/(1/k) &= k^2\lambda^{-\alpha} \leq \frac{\mu_6^2(\log n)^2}{\{\log((\kappa_n/t_n)\log n)\}^2} (\log n)^{-\alpha} \\ &= \frac{\mu_6^2}{\{\log((\kappa_n/t_n)\log n)\}^2 (\log n)^{\alpha-2}}. \end{aligned}$$

The right-hand side tends to zero if $\alpha \geq 2$. So $k\lambda^{-\alpha} < 1/k$ if n is large. Thus the measure of the exceptional set is at most

$$\frac{\mu_7}{k} \leq \mu \frac{\log\{(\kappa_n/t_n)\log n\}}{\log n}, \quad \text{if } \alpha \geq 2.$$

From the hypothesis, we get $\kappa_n/t_n < \log n$ for large n . Now suppose $1 \leq \alpha < 2$. Hence

$$\begin{aligned} (k\lambda^{-\alpha})/(1/k) &= k^2\lambda^{-\alpha} > \mu_5^2 \frac{(\log n)^{2-\alpha}}{\{\log((\kappa_n/t_n)\log n)\}^2} \\ &> \mu_5^2 \frac{(\log n)^{2-\alpha}}{\{\log(\log n)\}^2} = \frac{\mu_5^2}{4} \frac{(\log n)^{2-\alpha}}{(\log \log n)^2}. \end{aligned}$$

The right-hand side tends to infinity as n tends to infinity. Hence for $1 \leq \alpha < 2$, $1/k < k\lambda^{-\alpha}$ when n is large. Therefore the measure of the exceptional set is at most

$$\frac{\mu_6 \log n}{\{\log((\kappa_n/t_n)\log n)\}(\log n)^\alpha} = \frac{\mu_6 \log n}{\{\log((\kappa_n/t_n)\log n)\}(\log n)^{\alpha-1}} \quad \text{if } 1 \leq \alpha < 2.$$

3. THEOREM 2. Let $f(x) = a_0\xi_0 + a_1\xi_1x + a_2\xi_2x^2 + \cdots + a_n\xi_nx^n$ be a polynomial where ξ_v 's are independent random variables identically distributed with a common characteristic function $\exp(-C|t|^\alpha)$, C is a positive constant and $\alpha > 1$. $a_0, a_1, a_2, \dots, a_n$ are nonzero real numbers such that $\log(\kappa_n/t_n) = o(\log n)$ where κ_n and t_n have the same meaning as in Theorem 1. Then there exists an integer n_0 such that for each $n > n_0$ the number of real roots of $f(x) = 0$ is at least $\mu \log n / \log\{(\kappa_n/t_n) \log n\}$ outside a set of measure at most

$$\mu' / \left\{ \log \frac{\log n_0}{\log((\kappa_{n_0}/t_{n_0}) \log n_0)} \right\}^{\alpha-1}.$$

PROOF. In the proof of Samal and Mishra [4] we modify only the choice of M_m as follows and note the consequent changes in the different steps.

$$M_m = [\{2^\alpha A e((\kappa_n/t_n) \lambda_m)^\alpha\} / BM] + 1.$$

Then

$$\frac{\mu'_1 \log n}{\log\{(\kappa_n/t_n) \log n\}} < k < \frac{\mu'_2 \log n}{\log\{(\kappa_n/t_n) \log n\}}.$$

Hence $k \rightarrow \infty$ as $n \rightarrow \infty$. We take U_m, R_m as in §2.2 by replacing ξ_v by $a_v \xi_v$ and V_m by replacing ξ_v by $|a_v|^\alpha \xi_v$.

Our two lemmas shall be modified as follows.

LEMMA 2.2.

$$\left| \sum_{(2m+1)! M_m^{2m+1+1}}^n a_v \xi_v x_m^v \right| < \frac{1}{2} V_m$$

except for a set of measure at most

$$\left(\frac{\kappa_n}{t_n} \right)^\alpha \cdot \frac{2^{1+2\alpha} \cdot C}{1 + \alpha} \cdot \frac{Ae}{B} e^{-(2m-1)M_m},$$

for every sufficiently large m .

LEMMA 2.3.

$$\left| \sum_0^{(2m-1)! M_m^{2m-1}} a_v \xi_v x_m^v \right| < \lambda_m \left(\sum_0^{(2m-1)! M_m^{2m-1}} |a_v|^\alpha x_m^{\alpha v} \right)^{1/\alpha}$$

except for a set of measure at most $(2^{1+\alpha} \cdot C) / \{(1+\alpha) \lambda_m^\alpha\}$.

Proceeding as in Theorem 1, it will follow that $|R_n| < V_m$ except for a set of measure at most $\mu_1(\kappa_n/t_n)^\alpha e^{-(2m-1)M_m} + \mu_2 \lambda_m^{-\alpha}$, the rest of §2.4 will

remain unchanged. In §2.5,

$$E(\zeta_m) \leq \mu_1 \left(\frac{\kappa_n}{t_n} \right)^\alpha e^{-(2m-1)M_m} + \mu_2 \lambda_m^{-\alpha}.$$

We shall have $\sum_{m=m_0}^k \zeta_m < (k - m_0 + 1)\varepsilon_1$ outside an exceptional set of measure at most

$$\sum_{m=m_0}^k \frac{1}{(k - m_0 + 1)\varepsilon_1} \left\{ \mu_1 \left(\frac{\kappa_n}{t_n} \right)^\alpha e^{-(2m-1)M_m} + \mu_2 \lambda_m^{-\alpha} \right\} < \mu_3 \lambda_{m_0}^{-\alpha}.$$

This result is the same as that of the paper cited above since the choice of M_m was suitably modified. The remaining part of the proof remains unchanged except that we shall get

$$N_n > \mu \log n / \log\{(\kappa_n/t_n)\log n\} \quad \text{for all } n > n_0.$$

Also

$$m(G_{k_0}) < \mu' / \left\{ \log \left(\frac{\log n_0}{\log((\kappa_{n_0}/t_{n_0})\log n_0)} \right) \right\}^{\alpha-1}$$

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