WEAK EQUALS STRONG FOR FIRST-ORDER LINEAR PSEUDODIFFERENTIAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. Weak solutions are shown to be strong for a class of boundary value problems for first-order linear differential equations with zero order pseudodifferential matrix coefficients. The operators have the form $\chi A_j(x, D)\chi D_j\chi + \chi A_0\chi$, where χ is the characteristic function of a domain G . At the boundary, there are conditions on the coefficient A_n of normal differentiation, typically implying that $(\chi A_n)^{-1}$ is a bounded operator both on $L_2(G)$ and on $H_1(G)$. The proof uses tangential mollifiers and is also applied in a more abstract setting.

Introduction. In studying 'weak equals strong' for boundary value problems for first-order linear systems of partial differential or pseudodifferential operators ([1], [3], [4], [5], and others), one of the principal tools has been the use of tangential mollifiers as introduced in [3]. Their success usually depends on the fact that the system of equations can be reduced to an equivalent one of the same type for which the coefficient A_n of normal differentiation D_n commutes with a standard tangential mollifier.

K. O. Friedrichs has suggested a different type of system of pseudodifferential operators, namely one in which the characteristic function χ of the domain stands between the differential operators D_j and their pseudodifferential coefficients. Unless $A_n(x, D)$ has some special property however, such as satisfying the equation $\chi A_n\chi = \chi A_n$ (which would essentially reduce the system to a type considered by Agranovich [1]) or $\chi A_n\chi = A_n\chi$ (permitting one to multiply on the left by A_n^{-1} if it exists and considered by Sarason and Tartakoff [5]), one reduces the coefficient of D_n to one commuting with tangential mollifiers only by going outside the class of pseudodifferential operators. This procedure is carried out in §1 of the present paper. In §2, it is shown that strong convergence to zero of a sequence $[J_n, S]$ implies the same for sequences of the form $[J_n, f(S)]$, where f belongs to a large class of analytic functions.

Received by the editors July 3, 1972.

AMS (MOS) subject classifications (1970). Primary 35S15; Secondary 41A35.

Key words and phrases. Tangential mollifiers.

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1. Let $\Omega = x = (x', x_n): x_n < 0$ be a halfspace in R^n , and denote the boundary of Ω by Γ . We shall work with spaces of vector-valued functions with values in R^N , in particular $L_2(\Omega)$, $L_2(\Gamma)$, $H^1(\Omega)$, and $H_{1,0}(\Omega) = H_{1,0}$, the latter defined by its norm

$$\|u\|_{1,0}^2 = \|u\|^2 + \sum_{j \neq n} \|D_j u\|^2,$$

with the norms on the right-hand side above taken in $L_2(\Omega)$.

With χ the characteristic function of Ω , set

$$L = \sum \chi A_j(x, D) \chi D_j \chi + \chi A_0(x, D) \chi,$$

where each A_j is an N by N matrix of zero-order pseudodifferential operators of Kohn-Nirenberg type (we shall call such pseudodifferential operators ψ DO's, and denote by $\psi D(s)$ the space of ψ DO's of order s).

We assume that the leading symbol $A_n(x, \xi)$ of A_n can be written as a product of symbols

$$A_n(x, \xi) = \alpha(x, \xi) \beta(x, \xi) \mu(x),$$

where α , β and μ have uniformly bounded inverses, are constant for large x , and where, moreover,

(i) for fixed (x, ξ') with $|x_n|$ sufficiently small, $\alpha(x, \xi', \xi)$ is the boundary value of a function which is holomorphic in the upper half complex ξ_n -plane, and such that for each $\varepsilon > 0$,

$$\lim_{y \rightarrow +\infty} e^{-\varepsilon y} \sup_{\xi_n \text{ real}} |\alpha(x, \xi', \xi_n + i\varepsilon y)| = 0,$$

(ii) $\beta(x', 0, 0, 1) = \beta(x', 0, 0, -1)$, and

(iii) with some $k > 0$, $\text{Re}(\beta + \beta^*) \geq k$ uniformly.

It follows from (i), (ii), and (iii), that $A_n(x, D)$ can be written as

$$\alpha_0 \beta_0 \mu(x) + \gamma,$$

where α_0 and $\beta_0 \in \psi D(0)$ and differ from α and β by operators of order minus one, γ has order minus one, and where

(iv) $\chi \alpha_0 = \chi \alpha_0 \chi$,

(v) $\chi(\beta + \beta^*) \chi \geq k$ as an operator on $L_2(\Omega)$, and

(vi) $\chi \alpha_0$ and $\chi \beta_0$, considered as operators either from $L_2(\Omega)$ to itself or from $H^1(\Omega)$ to itself, are bounded and have bounded inverses, and $\chi \beta_0$ has a bounded inverse from $H_{1,0}$ to itself (see [5]).

We shall study first the equivalence of appropriately defined weak and strong solutions of the boundary value problem

$$(1) \quad \begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= \bar{u} \quad \text{on } \Gamma, \end{aligned}$$

where $f \in L_2(\Omega)$ and $\bar{u} \in L_2(\Gamma)$.

Without loss, we assume that $\gamma=0$. This can be done because for functions $u \in H^1(\Omega)$, $\chi\gamma\chi D_n\chi u$ is a bounded function of the pair (u, u_Γ) , where u_Γ is the boundary value of u . In dealing with the more general boundary condition $Pu=0$, we must make the additional assumption that $\chi\gamma\delta(x_n)\otimes(1-P)$ is a bounded map from $L_2(\Gamma)$ to $L_2(\Omega)$, as it will be, e.g. if $\gamma \in \psi D(-2)$.

In defining weak solutions of (1), we take advantage of the fact that Green's formula is known (see [2], [5]) in the case that $\alpha_0=I$. We shall say that the pair $(u, \bar{u}) \in L_2(\Omega) \times L_2(\Gamma)$ is a weak solution of (1) provided

$$(1') \quad \begin{aligned} (\chi\alpha_0)^{-1}Lu &= (\chi\alpha_0)^{-1}f && \text{in } \Omega, \\ u &= \bar{u} && \text{on } \Gamma \end{aligned}$$

weakly in a sense to be defined below. This has the effect of taking the pre-closed domain of the adjoint operator to be the set $\chi\alpha_0^*\chi H^1(\Omega)$.

Thus we can and do assume without loss that $\alpha=I$. We say that the pair $(u, \bar{u})$ is a weak solution of (1) if it satisfies Green's formula, namely

$$(2) \quad (f, v) - (u, L^*v) = i(u, \mu^*\beta^*v)_\Gamma$$

for all $v \in H^1(\Omega)$. We shall say that $(u, \bar{u})$ is a strong solution of (1) provided there is a sequence $u_j \in H^1(\Omega)$ such that

$$(3) \quad \|u_j - u\|^2 + \|Lu_j - f\|^2 + \|u_j - \bar{u}\|_\Gamma^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

THEOREM 1. $U=(u, \bar{u})$ is a weak solution of (1) if and only if U is a strong solution.

PROOF. That strong solutions are weak solutions is known. To prove that weak solutions are strong, we shall use tangential mollification.

Because β^* satisfies the same conditions as β , we have that $\chi\beta\chi$, $\chi\beta^*\chi$, μ and μ^* are all bounded operators, both on $L_2(\Omega)$ and on $H^1(\Omega)$. In particular, it thus suffices to use in (2) test functions of the form $(\chi\beta^*)^{-1}(\mu^*)^{-1}v$ with $v \in H^1(\Omega)$; in this case (2) becomes

$$(2') \quad (\mu^{-1}(\chi\beta)^{-1}f, v) - (u, L^*(\chi\beta^*)^{-1}(\mu^*)^{-1}v) = i(\bar{u}, v)_\Gamma.$$

We wish to show that there exists a sequence $\{J_m\}$ of tangential mollifiers converging to I and which has the property that the commutators

$$(4) \quad [J_m, L^*(\chi\beta^*)^{-1}(\mu^*)^{-1}]$$

tend strongly to zero in $L_2(\Omega)$. The existence of normal derivatives of the mollified functions J_mu and convergence of their boundary values to $\bar{u}$ can be shown routinely, e.g., as in [4].

REMARK 1. Essentially the same argument (see Lax and Phillips [3]) shows that weak equals strong for solutions of homogeneous boundary value problems with a boundary condition of the form $Pu=0$, where P is a bounded operator on $L_2(\Gamma)$ which commutes with translations. In this case the defining relation for weak solutions takes the form

$$(2'') \quad ((\chi\beta\mu)^{-1}f, v) - (u, L^*(\chi\mu^*\beta^*)^{-1}v) = 0$$

for all v such that $(1-P^*)v=0$ on Γ . For u to be a strong solution in this set-up, the approximating functions $J_m u$ should satisfy $PJ_m u=0$ on Γ , which they will provided J_m commutes with P .

We take for J_m the operator $(J_m u(x)) = m^{n-1} \int j(n(x'-y'))u(y', x_n) dy'$, where $j \in C_0^\infty(R^{n-1})$ has average value one. The sequence of operators $\{J_m\}$ has the following properties:

(a) Each J_m is a contraction and the sequence $\{J_m\}$ converges strongly to the identity on any of the spaces $L_2(\Omega)$, $L_2(\Gamma)$, $H^1(\Omega)$ and $H_{1,0}$.

(b) $[J_m, D_j] = [J_m, \chi] = 0$.

(c) If C is any zero-order ψ DO with a symbol compactly supported in x , then the sequences

$$[J_m, C], D_j[J_m, C] \quad \text{and} \quad [J_m, C]D_j \quad (j \neq n)$$

are uniformly bounded and converge to zero strongly as operators on $L_2(\Omega)$ and on $H_{1,0}$.

To prove (4), it suffices to show that with all zero-order ψ DO's C whose symbols have compact support in x ,

$$(5) \quad [J_m, \chi D_j C (\chi\mu^*\beta^*)^{-1}] \rightarrow 0 \quad \text{strongly in } L_2(\Omega), \quad j \neq n.$$

To help prove (5), we shall need the following lemma.

LEMMA 1. *As operators from $L_2(\Omega)$ to $H_{1,0}$, $[J_m, \chi C]$ and $[J_m, \chi\beta^*\chi]$ are uniformly bounded in m and tend to zero strongly as $m \rightarrow \infty$.*

Lemma 1 is an immediate consequence of the listed properties of the sequence $\{J_m\}$.

To prove (5), we first write

$$\begin{aligned} & [J_m, \chi D_j (\chi\mu^*\beta^*)^{-1}] \\ (6) \quad &= \chi D_j [J_m, \chi C] (\chi\mu^*\beta^*)^{-1} + \chi D_j C [J_m, (\chi\beta^*)^{-1}] \mu^{*-1} \\ &= \chi D_j C (\chi\beta^*)^{-1} [J_m, \mu^{*-1}]. \end{aligned}$$

The first and third terms on the right-hand side of (6) are easily seen to converge to zero strongly; because of Lemma 1 and the fact that χC is a bounded operator on $H_{1,0}$, it suffices to show that the sequence $[J_m, (\chi\beta^*)^{-1}]$ tends strongly to zero.

The proof is by a homotopy argument. Since $\chi\beta^*\chi$ has a positive symmetric part, the operators

$$S_t = (1 - t)I + t\chi\beta^*\chi, \quad 0 \leq t \leq 1,$$

have uniformly positive symmetric parts, and thus have inverses T_t which are uniformly bounded. Further, we have

LEMMA 2. *The operators $T_t: H_{1,0} \rightarrow H_{1,0}$ are uniformly bounded for $0 \leq t \leq 1$.*

PROOF. We claim that $\beta + \beta^*$ is a positive operator on $H_{1,0}$. To show this we use the equivalent norm $\|u\|_{\varepsilon}$ defined as

$$\|u\|_{\varepsilon} = \left(\|u\|_{\Omega}^2 + \varepsilon^2 \sum_{j \neq n} \|D_j u\|^2 \right)^{1/2}.$$

If $u \in H_{1,0}$, then

$$\begin{aligned} (u, \beta u)_{\varepsilon} &= (u, \beta u) - \varepsilon^2 \sum_{j \neq n} (u, D_j^2 \beta u) \\ &= (u, \beta u) + \varepsilon^2 \sum_{j \neq n} (D_j u, \beta D_j u) + \varepsilon^2 \sum_{j \neq n} (D_j u, [\beta, D_j] u) \\ &\geq \frac{k}{2} \|u\|_{\varepsilon}^2 - p \varepsilon^2 \sum_{j \neq n} \|D_j u\| \|u\|, \end{aligned}$$

where $p = \sup_{j \neq n} \|\beta, D_j\|$. Thus if $0 < \varepsilon < (2pn)^{-1}$, the form on the right is positive. Lemma 2 is proved.

The following lemma will conclude the proof of Theorem 1.

LEMMA 3. *Let $\Lambda = (1 - \sum_{j \neq n} D_j^2)^{1/2}$. There exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ and if $\Lambda[J_m, T_t] \rightarrow 0$ strongly on $L_2(\Omega)$, then $\Lambda[J_m, T_{t+\delta}] \rightarrow 0$ strongly as $m \rightarrow \infty$ with $0 \leq t \leq t + \varepsilon \leq 1$.*

Since $T_0 = I$ commutes with J_m , the lemma ensures that $\Lambda[J_m, T_1] = \Lambda[J_m, (\chi\beta^*)^{-1}] \rightarrow 0$ strongly as $m \rightarrow \infty$.

PROOF OF LEMMA 3. Set $\gamma = \chi(\beta^* - I)\chi$, and let

$$M = \max(\|\gamma\|_{L_2(\Omega)}, \|\gamma\|_{H_{1,0}}) \cdot \sup_{0 \leq t \leq 1} (\max \|T_t\|_{L_2(\Omega)}, \|T_t\|_{H_{1,0}}).$$

If $0 < \delta_0 < (2M)^{-1}$, we have, when $0 < \delta < \delta_0$ and $t + \delta \leq 1$,

$$\begin{aligned} \Lambda[J_m, T_{t+\delta}] &= \sum_{k=0}^{\infty} \Lambda[J_m, \delta^k (T_t \gamma)^k T_t] \\ &= \sum_{k=0}^{\infty} \Lambda[J_m, \delta^k (T_t \gamma)^k] T_t + \Lambda(1 + \delta T_t \gamma)^{-1} [J_m, T_t]. \end{aligned}$$

The sequence $\Lambda(1 + \delta T_i \gamma)^{-1} [J_m, T_i]$ converges strongly to zero because of Lemmas 1 and 2. There remains to show the strong convergence to zero of $\sum [J_m, \delta^k (T_i \gamma)^k]$, which we expand as

$$\delta \left(\sum_{k=1}^K + \sum_{k=K+1}^{\infty} \right) \Lambda \left(\sum_{r=s=1=k} (\delta T_i \gamma)^r [J_m, T_i \gamma] (\delta T_i \gamma)^s \right).$$

The sum $\sum_{K+1}^{\infty}$ can be estimated in norm by

$$\delta \sum_{K+1}^{\infty} k (\delta M)^{k-1} \| [J_m, T_i \gamma] \| = \delta \| [J_m, T_i \gamma] \| \frac{d}{dy} \frac{y^{K+1}}{1-y} \Big|_{y=\delta M},$$

which tends to zero as $K \rightarrow \infty$. Since the finite sum $\sum_0^K$ converges strongly to zero as $m \rightarrow \infty$, so also does the full sum.

REMARK 2. Because the operator L differs from a sum of operators with support in coordinate patches by an operator of order minus infinity, Theorem 1 extends immediately to first-order linear operators of the same type on compact manifolds with boundary.

2. The argument given in §1 suggests a more general result on mollifiers which we now give, but in a more abstract setting.

Let B and B_1 be Banach spaces with norms denoted by $|\cdot|$ and by $\|\cdot\|$ respectively such that B_1 is imbedded in B and has the stronger topology—in particular, let $|u| \leq \|u\|$ for all $u \in B_1$. Let S be a bounded linear operator on B mapping B_1 into itself and whose restriction to B_1 , considered as an operator on B_1 and denoted by S_1 , is also bounded. Denote the larger of the norms of S and of S_1 by $\langle S \rangle$, and denote the union of the spectra of S and of S_1 by σ .

Suppose that $\{J_n\}$, $n=1, 2, \dots$, is a sequence of linear operators from B into B_1 , each bounded in that sense, converging strongly to the identity as operators from either B or B_1 to itself, and such that the commutators $[J_n, S]$, considered as operators from B into B_1 , are uniformly bounded:

$$\| [J_n, S] u \| \leq M |u|, \quad u \in B,$$

and converge strongly to zero.

THEOREM 2. Let f be a function of one complex variable which is analytic on σ . Suppose that there exists a smooth simple closed Jordan curve C such that σ lies in the component of the plane interior to C , and such that f is analytic on the interior component and on a neighborhood of C . Then the sequence $\{[J_n, f(S)]\}: B \rightarrow B_1$ tends strongly to zero.

PROOF. Writing $f(S)$ as a contour integral

$$(7) \quad f(S) = (2\pi i)^{-1} \oint_C (z - S)^{-1} f(z) dz,$$

we need information about the commutators $[J_n, (z-S)^{-1}]$, and this information is furnished by the following lemma.

LEMMA 4. *Suppose that zero lies in the unbounded component U of the complement of σ . Then the commutators $[J_n, S^{-1}]$ tend strongly to zero as $n \rightarrow \infty$, and the quantity*

$$b = \sup_n \sup_{u \in B} \|[J_n, S^{-1}]u\|/|u|$$

is finite and can be estimated from a knowledge only of $M, \langle S \rangle$, and the length l and the minimum distance d to σ of some path γ lying entirely in U and which begins at 0 and ends at a point z_1 of modulus $2\langle S \rangle$. The estimate is improved by decreasing l or by increasing d .

PROOF OF LEMMA 4. The proof is an extension of the method of the proof of Lemma 3. We start by using the method of the proof of Lemma 3 to estimate $[J, (z_1 - S)^{-1}] = z_1^{-1}[J, (1 - z_1^{-1}S)^{-1}]$. Next we select a point z_2 on γ at a distance $d/2$ from z_1 (or set $z_2 = 0$ if $|z_1| < d/2$). We want to estimate $[J, (z_2 - S)^{-1}]$, which we write as

$$[J, (1 + (z_2 - z_1)(z_1 - S)^{-1})(z_1 - S)^{-1}]$$

and which equals

$$[J, (1 + (z_2 - z_1)(z_1 - S)^{-1})(z_1 - S)^{-1}] \\ + (1 + (z_2 - z_1)(z_1 - S)^{-1})^{-1}[J, (z_1 - S)^{-1}].$$

The second term on the right-hand side above has in effect already been estimated, while to estimate the first, we again use the method of the proof of Lemma 3. Moving along γ in this way and estimating at each step $\langle (z_i - S)^{-1} \rangle \leq d^{-1}$, we arrive after a finite number of steps at an estimate for $[J, S^{-1}]$. That the estimate depends only on the listed quantities and in the prescribed way, is easily checked by the reader. Strong convergence of the sequence $\{[J_n, S^{-1}]\}$ to zero follows as in the proof of Lemma 3.

We can now conclude the proof of Theorem 2. Because of Lemma 4, the commutators $[J_n, (z-S)^{-1}]$ are bounded independently of n and of z for z on C . Further, one sees easily from the representation

$$[J, (z - S)^{-1}] - [J, (z' - S)^{-1}] \\ = (z' - z)(([J, (z - S)^{-1}](z' - S)^{-1} + (z - S)^{-1}[J, (z' - S)^{-1}])$$

that these commutators are, for z on C , uniformly Lipschitz continuous with Lipschitz constant independent of n .

It follows that the sequence of operators

$$[J_n, f(S)] = (2\pi i)^{-1} \int_C [J_n, (z - S)^{-1}] f(z) dz$$

can be approximated, uniformly in norm, by sequences of Riemann sums of the form

$$(2\pi i)^{-1} \sum_{j=1}^N [J_n, (z_j - S)^{-1}] f(z_j)(z_j - z_{j-1})$$

with z_j a fixed point on C , $j=1, \dots, N$. Since the sequences $[J_n, (z_j - S)^{-1}]$ converge strongly to zero, so also do the sequences of Riemann sums, and hence also their limits $[J_n, f(S)]$. Q.E.D.

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