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A CLASS OF PARTIALLY ORDERED LINEAR ALGEBRAS

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ABSTRACT. We consider a special type of partially ordered linear algebra which is like an algebra of real-valued functions. We show that various natural properties characterize this type of algebra. These natural properties relate the algebraic and order structures to each other.

A *pola* (denoted by A) is a real linear associative algebra which is partially ordered so that it is a directed partially ordered linear space and $0 \leqq xy$ whenever $x, y \in A$, $0 \leqq x$, $0 \leqq y$. We also assume that A has a multiplicative identity $1 \geqq 0$. A Dedekind σ-complete pola (dsc-pola) A is one having the property: if $x_n \in A$, $0 \leqq \cdots \leqq x_2 \leqq x_1$, then $\inf\{x_n\}$ exists. Order convergence is defined as usual. A dsc-pola A has the Archimedean property: if $x, y \in A$ and $nx \leqq y$ for every positive integer n, then $x \leqq 0$. For more details and examples see the references.

The simple example of interest to us here is the dsc-pola A of all real-valued functions defined on some nonempty set, where the algebraic operations and the partial order are defined pointwise. We note that A has the following property:

P_1: If $x \in A$ and $x \geqq 1$, then x has an inverse and $x^{-1} \geqq 0$.

If we now consider an arbitrary dsc-pola A which has property P_1, then we can show that A is much like an algebra of real-valued functions; however, the operations may only be defined "almost everywhere" (see example 5 of [**4**]). Some of the basic properties are given in the following theorem.

THEOREM 1. *If A is a dsc-pola which has property* P_1, *then multiplication of elements in A is commutative and A is a lattice. Furthermore, $x^2 \geqq 0$ for all $x \in A$ and if $y \in A$ and $y \geqq 0$, then there exists a unique $z \in A$ such that $z^2 = y$ and $z \geqq 0$.*

This theorem was proved by the author but it appears as the necessary introduction to the thesis of his former student, T. Dai, who showed in

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addition that A (having property P_1) is an f-ring [**1**, p. 403]. The reader is referred to Dai's paper [**2**] for examples and proofs. The purpose of this paper is to show that various natural properties for a dsc-pola A imply that A has property P_1.

LEMMA 1. *Let A be a dsc-pola. If $x \leqq 1$ and there exists $y \geqq 0$ such that $1 \leqq xy$ or $1 \leqq yx$, then x has an inverse and $x^{-1} \geqq 1$. From this it follows that if $1 \leqq u \leqq v$ and v has an inverse and $v^{-1} \geqq 0$, then u has an inverse and $u^{-1} \geqq 0$.*

For the proof see Proposition 3 and its corollary in [**3**].

LEMMA 2. *Let A be a dsc-pola which has the property: if $x \in A$ and $x \geqq 0$, then there exists a sequence $\{x_n\}$ of elements from A such that $0 \leqq x_n \leqq n1$ for all n and o-$\lim x_n = x$. Then A has property P_1.*

For the proof see (v) of Lemma 1.6.6 in [**2**].

The following two properties concern one-sided factoring of one element by another.

THEOREM 2. *Let A be a dsc-pola which has the property: if $y_1, y_2 \in A$ and $0 \leqq y_1 \leqq y_2$, then there exists $w \in A$ such that $w \geqq 0$ and $y_1 = wy_2$. Then A has property P_1.*

PROOF. Take any $x \in A$ such that $x \geqq 1 \geqq 0$. Hence, there exists $w \in A$ such that $w \geqq 0$ and $wx = 1$. Since $w \geqq 0$ and $x \geqq 1$, we have $w \leqq 1$. Using Lemma 1, we see that $x^{-1} = w \geqq 0$.

THEOREM 3. *Let A be a dsc-pola which has the property: if $y_1, y_2 \in A$ and $1 \leqq y_1 \leqq y_2$, then there exists $w \in A$ such that $w \geqq 1$ and $wy_1 = y_2$. Then A has property P_1.*

PROOF. Take any $x \in A$ such that $x \geqq 1$. Since $1 \leqq x \leqq x+1$, there exists $w \in A$ such that $w \geqq 1$ and $wx = x+1$. Hence, $(w-1)x = 1$, where $w-1 \geqq 0$. Thus, $w-1 \leqq 1$ and we may again use Lemma 1 to show that $x^{-1} = w-1 \geqq 0$.

The following is a decomposition property for multiplication in a dsc-pola which is commutative.

THEOREM 4. *Let A be a commutative dsc-pola which has the property: if $y \in A$, $y \geqq 0$ and $0 \leqq w \leqq y^2$, then there exist elements $u, v \in A$ such that $0 \leqq u \leqq y$, $0 \leqq v \leqq y$ and $uv = w$. Then A has property P_1.*

PROOF. Take any $x, y \in A$ such that $1 \leqq x \leqq x+1 \leqq y$. Thus, $0 \leqq y^2 - 1 \leqq y^2$. Hence, we may find $u, v \in A$ such that $0 \leqq u \leqq y$, $0 \leqq v \leqq y$, and $uv = y^2 - 1$. We see easily that $1 = y(y-u) + u(y-v)$. We remark that this is the only place we use commutativity. Using the inequalities given above, one can easily show that $0 \leqq y-u \leqq 1$, $u \geqq 1$ and then $0 \leqq y-v \leqq 1$. Since

$1+(y-u)(y-v)=y(2y-u-v)$ and $0\leqq(y-u)(y-v)\leqq 1$, we see that $(\frac{1}{2})y(2y-u-v)\leqq 1$ and then by using Lemma 1, we obtain $[y(2y-u-v)]^{-1}\geqq 0$. By using Lemma 1 twice, one can show first that $y^{-1}\geqq 0$ and then $x^{-1}\geqq 0$.

At the end of the paper a counterexample will be given to show that A must be commutative in the previous theorem. However, we can drop commutativity if we use a stronger decomposition property as follows.

THEOREM 5. *Let A be a dsc-pola which has the property: if $y\in A$, $y\geqq 0$ and $0\leqq w\leqq y^2$, then there exists $u\in A$ such that $0\leqq u\leqq y$ and $u^2=w$. Then A has property* P_1.

PROOF. Take any x, $y\in A$ such that $1\leqq x\leqq x+1\leqq y$. Thus, $0\leqq y^2-1\leqq y^2$. Hence, we may find $u\in A$ such that $0\leqq u\leqq y$ and $u^2=y^2-1$. We see easily that $1=y(y-u)+(y-u)u$. Using the inequalities given above, one can easily show that $0\leqq y-u\leqq 1$. Since $0\leqq y-u$, we see that $y(y-u)\leqq 1$ and $y(y-u)(1+u)=y(y-u)+y(y-u)u\geqq 1$. Using Lemma 1 twice, we can first show that $[y(y-u)]^{-1}\geqq 0$ and then $y^{-1}\geqq 0$. Using Lemma 1 again, we can show that $x^{-1}\geqq 0$.

Next we consider an order-reversing property for left inverses.

THEOREM 6. *Let A be a dsc-pola which has the property: if y_1, $y_2\in A$ and $1\leqq y_1\leqq y_2$, then there exist w_1, $w_2\in A$ such that $w_2\leqq w_1$ and $w_1y_1=w_2y_2=1$. Then A has property* P_1.

PROOF. Take any $x\in A$ such that $1\leqq x\leqq x+1$. There exist w_1, $w_2\in A$ such that $w_2\leqq w_1$ and $w_1x=w_2(x+1)=1$. Hence, $(w_1-w_2)x=w_2$ and $(w_1-w_2)x(x+1)=1$. Since $x(x+1)\geqq 1$ and $0\leqq w_1-w_2$, we have $w_1-w_2\leqq 1$. Using Lemma 1 twice, we can first show that $[x(x+1)]^{-1}=w_1-w_2\geqq 0$ and then $x^{-1}\geqq 0$.

The next property concerns generalized inverses.

THEOREM 7. *Let A be a dsc-pola which has the property: if $z\in A$ and $z\geqq 1$, then there exists $w\in A$ such that $w\geqq 0$ and $zwz=z$. Then A has property* P_1.

PROOF. From the above it follows that if $v\in A$, $v\geqq 1$ and v has an inverse, then $v^{-1}\geqq 0$. Let us now take w and z as in the statement of the theorem. If we put $u=wz$, then $0\leqq u=u^2$. Since $1+nu\geqq 1$ and $1+nu$ has an inverse for every positive integer n, we obtain $0\leqq(1+nu)^{-1}=1-(n/n+1)u$ for all n. Using the Archimedean property, we obtain $wz=u\leqq 1$. Since $zwz=z\geqq 1$, we obtain $(wz)^{-1}\geqq 1$ by using Lemma 1. By again using Lemma 1 we obtain $z^{-1}\geqq 0$.

The following question is unanswered.

Question. Let A be a dsc-pola which has the property: if $x \in A$, then $x^2 \geqq 0$. Does A have property P_1?

We can answer this question in certain special cases.

THEOREM 8. *Let A be a dsc-pola which has the properties: A is a lattice and if $x \in A$, then $x^2 \geqq 0$. Then A has property* P_1.

PROOF. Take any $z \in A$ such that $z \geqq 0$. Since $(2n1-z)^2 \geqq 0$, we get $0 \leqq z \leqq n1+(1/4n)z^2$ for every positive integer n. Since A is a lattice, we may write $z=z_n+w_n$, where $0 \leqq z_n \leqq n1$ and $0 \leqq w_n \leqq (1/4n)z^2$ for all n. Thus, o-$\lim z_n = z$. Using Lemma 2, we see that A has property P_1.

Some definitions are necessary for the next two theorems. An element $u \in A$ is called an order unit if $u \geqq 0$ and if for any $x \in A$ there exists a real number α such that $-\alpha u \leqq x \leqq \alpha u$. A dsc-pola A is said to have the Perron-Frobenius (PF) property if for every $x \in A$, $x \geqq 0$, there exists a real number $\lambda > 0$ such that $\lambda 1 - x$ has an inverse and $(\lambda 1 - x)^{-1} \geqq 0$. The name of this property is justified in **[4]**. A dsc-pola A is said to have the large inverse property if for any $x \in A$ there exists $y \in A$ such that $x \leqq y$ and y has an inverse. The large inverse property plays a key role in Theorem 10. The next theorem gives a chain of implications showing that the large inverse property is a consequence of other natural properties.

THEOREM 9. *Let A be a dsc-pola. If A is finite-dimensional, then A has an order unit. If A has an order unit, then A has the* PF *property. If A has the* PF *property, then A has the large inverse property.*

PROOF. The first implication is a consequence of two facts: A is finite dimensional and A is directed. The proof of the second implication can be found in Theorem 6 of **[4]**. The proof of the third implication can be found in Proposition 3 of **[3]**.

THEOREM 10. *Let A be a dsc-pola which has the large inverse property and also has the property: if $z \in A$, then $z^2 \geqq 0$. Then A has property* P_1.

PROOF. Take any $x \in A$ such that $x \geqq 1$. Since A has the large inverse property, there exists $y \in A$ such that $x \leqq y$ and y has an inverse. From the second property we obtain $0 \leqq (y^{-1})^2 = (y^2)^{-1}$. Since $1 \leqq x \leqq y \leqq y^2$, we can use Lemma 1 to show that $x^{-1} \geqq 0$.

The final two theorems concern special assumptions about the way an element can be expressed as the difference of two nonnegative elements.

THEOREM 11. *Let A be a dsc-pola which has the property: if $x \in A$, then there exists $a \in A$ such that $0 \leqq a \leqq 1$, $ax \geqq 0$ and $(1-a)x \leqq 0$. Then A has property* P_1.

PROOF. Take any $z \in A$ such that $z \geqq 0$. Next select $a_n \in A$ such that $0 \leqq a_n \leqq 1$, $a_n(n1-z) \geqq 0$ and $(1-a_n)(n1-z) \leqq 0$ for every positive integer n. Hence, $0 \leqq a_n z \leqq n a_n \leqq n1$ and $0 \leqq n(1-a_n) \leqq (1-a_n)z$. From the latter inequalities and the fact that $0 \leqq 1-a_n \leqq 1$, we obtain $0 \leqq n(1-a_n) \leqq z$ and then $0 \leqq z-a_n z \leqq (1/n)z^2$. Putting $z_n = a_n z$, we see that $0 \leqq z_n \leqq n1$ and o-$\lim z_n = z$. We may now use Lemma 2 to show that A has property P_1.

The last theorem was proved by the author but credit is due Ralph Gellar. He proved a slightly weaker theorem which inspired the author to work on the following theorem.

THEOREM 12. *Let A be a dsc-pola which has the property: if $x \in A$, then there exist y, $z \in A$ such that $y \geqq 0$, $z \geqq 0$, $yz=0$ and $x=y-z$. Then A has property P_1. (Gellar also assumed that $zy=0$.)*

PROOF. The key idea involved is that if $a \in A$ and $0 \leqq a^2 \leqq a$, then $a \leqq 1$. We first prove this fact.

There exist b, $c \in A$ such that $b \geqq 0$, $c \geqq 0$, $bc=0$ and $1-a=b-c$. Since $a^2 \leqq a$, we have $0 \leqq a-a^2 = a(1-a) = a(b-c)$, which means that $0 \leqq ac \leqq ab$. Hence, $0 \leqq ac^2 \leqq abc = 0$, which means that $ac^2=0$. Since $1 \leqq 1+c = a+b$, we obtain $0 \leqq c^2 \leqq ac^2 = 0$, which means that $c^2=0$. Now there exist d, $e \in A$ such that $d \geqq 0$, $e \geqq 0$, $de=0$ and $1-c=d-e$. Therefore, $1 \leqq 1+e = c+d$ so that $e \leqq ce$. Hence, $0 \leqq e \leqq ce \leqq c^2 e = 0$, which means that $e=0$. It follows that $0 \leqq 1-c$ so that $0 \leqq (1-c)^n = 1-nc$ for every positive integer n. From the Archimedean property it follows that $c=0$, which means that $a \leqq 1$.

Now take any $h \in A$ such that $h \geqq 0$. For each positive integer n there exist y_n, $z_n \in A$ such that $y_n \geqq 0$, $z_n \geqq 0$, $y_n z_n = 0$ and $n1-h = y_n - z_n$. Hence, $0 \leqq y_n^2 \leqq y_n(y_n+h) = y_n(n1+z_n) = ny_n$ for all n. Thus, $0 \leqq (1/n)^2 y_n^2 \leqq (1/n) y_n$ so that $0 \leqq y_n \leqq n1$ for all n. The last inequality is obtained from the result of the preceding paragraph. Since $y_n \leqq n1$, we obtain $z_n \leqq h$ for all n. Now $nz_n \leqq (n1+z_n)z_n = (y_n+h)z_n = hz_n \leqq h^2$ for all n, which means that if we define $h_n = n1 - y_n$, then $0 \leqq h - h_n \leqq (1/n)h^2$. Thus, $0 \leqq h_n \leqq n1$ and o-$\lim h_n = h$. Using Lemma 2, we see that A has property P_1.

COUNTEREXAMPLES. Let M be the real linear algebra of all 2-by-2 matrices in upper triangular form, where all entries are real. If M is partially ordered entry by entry, then M is a dsc-pola which is not commutative. The reader is invited to use M to verify that the order conditions are necessary in Theorems 2, 3, 5, 6 and 7. For example, look at the proof of Theorem 2. If we take any $x \in M$ such that $x \geqq 1$, then there exists $w \in M$ such that $wx=1$, but it may happen that w not $\geqq 0$. The dsc-pola M has the property described in Theorem 4 but it is not commutative. Note that M is a lattice and has the large inverse property but it does not have the other property needed in Theorems 8 and 10. Also M does not have the properties described in Theorems 11 and 12.

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