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INDECOMPOSABLE CONTINUA IN STONE-ČECH COMPACTIFICATIONS

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ABSTRACT. We show that if Y is a continuum irreducible from a to b, which is connected im Kleinen and first countable at b, and if $X=Y-\{b\}$, then $\beta X-X$ is an indecomposable continuum. Examples are given showing that both first countability and connectedness im Kleinen are needed here. We also show that $\beta[0, 1)-[0, 1)$ has a strong near-homogeneity property.

1. **Introduction.** In [**2**] and [**3**] it is shown that if $X=[0, 1)$ then $\beta X-X$ is an indecomposable continuum; here βX is the Stone-Čech compactification of X. In [**7**], Dickman showed that among locally connected spaces, $[0, 1)$ is essentially the only such space. In this paper we exhibit other types of spaces X with this property. We shall also show that for $X=[0, 1)$, $\beta X-X$ is stably almost homogeneous, a concept to be defined below.

The set function T has been studied and applied in [**1**], [**5**], [**6**], [**8**], [**9**], [**11**], and [**14**]. We follow these papers in writing $T(p)$ for $T(\{p\})$. This set function will be used in the argument at one point and familiarity with it is assumed. Familiarity with [**10**], [**12**], [**13**], and [**15**] is also assumed. If we write $X=A\cup B$ sep, then we mean that $\mathrm{Cl}(A)\cap B=\varnothing$ and $A\cap \mathrm{Cl}(B)=\varnothing$ while neither A nor B is empty. By $f: X\simeq Y$, we mean f is a homeomorphism of X onto Y.

2. **Indecomposable continua in βX.**

LEMMA 1. *There is a covariant functor β on the category of Tychonoff spaces and continuous maps such that for any Tychonoff space X, βX is the Stone-Čech compactification of X and if $f: X\to Y$ then $\beta f: \beta X\to\beta Y$ is the unique extension of f induced by f treated as a map from X to βY.*

Notation. If X is a Tychonoff space, let $\gamma X=\beta X-X$. If f is a continuous map from X to Y, let γf denote $\beta f|\gamma X$.

DEFINITION. Let Y be a compact Hausdorff continuum irreducible from a to b such that Y is both connected im Kleinen and first countable

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at b. Let $X=Y-\{b\}$. Then we call the topological pair (Y, X) a *wave* from a to b.

By stringing together indecomposable continua, a wave (Y, X) can be constructed such that Y is not connected im Kleinen at any point of X.

LEMMA 2. *If Y is a compact Hausdorff continuum irreducible from a to b and $x \in Y$, $T(x)$ either separates a from b, contains a, or contains b. In case $T(x)$ separates a from b, $Y-T(x)$ has exactly two components, A and B, where $a \in A$ and $b \in B$, and both $T(x) \cup A$ and $T(x) \cup B$ are proper subcontinua of Y containing a and b respectively.*

REMARK ON PROOF. This lemma can be established using standard techniques and Theorem 1.10 of [**14**], since each $x \in Y$ different from a and b weakly separates a from b.

LEMMA 3. *If Y is a compact Hausdorff continuum irreducible from a to b and $W \subseteq Y$ is a continuum with $b \in \operatorname{Int} W$, then $W-\{b\}$ is connected.*

PROOF. If $W-\{b\}=M_0 \cup N_0$ sep, let $M=M_0 \cup \{b\}$; $N=N_0 \cup \{b\}$. Then b lies in the boundary of M and N and, by Theorem 6 of [**15**, p. 194], each of M and N is nowhere dense, so that $M \cup N=W$ is nowhere dense also, a contradiction.

LEMMA 4. *If (Y, X) is a wave from a to b, and Z is a Hausdorff compactification of X, then $Z-X$ is a Hausdorff continuum.*

PROOF. Since Y is connected im Kleinen and first countable at b, there exists a denumerable collection of continua $\{N_i\}_{i=1}^{\infty}$ such that for each i, $b \in \operatorname{Int}(N_i)$ and $N_{i+1} \subseteq N_i$ and $\bigcap_{i=1}^{\infty} N_i=\{b\}$. It is readily seen that

$$Z - X = \operatorname{Cl}_Z(N_i - \{b\}) - N_i = \bigcap_{i=1}^{\infty} \operatorname{Cl}_Z(N_i - \{b\}).$$

Then, since each $N_i-\{b\}$ is connected by Lemma 3, $Z-X$ is an intersection of a monotone collection of continua.

LEMMA 5. *Let X be a compact Hausdorff space, $b \in X$, $\{b\}$ a component of X, and suppose X is first countable at b and $\langle b_i \rangle_{i=1}^{\infty}$ is a sequence in $X-\{b\}$ converging to b. Then there exist two closed subsets A and B of X such that $A \cup B=X$, $A \cap B=\{b\}$, and each of A and B contains infinitely many (that is, a subsequence) of the b_i's.*

PROOF. It is readily seen that there exists a neighborhood basis $\{N_j\}_{j=1}^{\infty}$ at b consisting of closed and open sets such that $N_{j+1} \subseteq N_j$ for each j and $N_1=X$; by passing to a subset if necessary, we may suppose that each

$N_j - N_{j+1}$ contains at least one of the b_i's. Then set

$$A = \{b\} \cup \bigcup_{j=1}^{\infty} (N_{2j-1} - N_{2j}), \qquad B = \{b\} \cup \bigcup_{j=1}^{\infty} (N_{2j} - N_{2j+1}).$$

Then A and B have the desired properties.

LEMMA 6. *If (Y, X) is a wave from a to b and W is a nondegenerate subcontinuum of Y containing b, then $b \in \operatorname{Int} W$.*

PROOF. Suppose not. Then let $p \in W$, $p \neq b$. Since $b \notin T(p)$, by connectedness im Kleinen at b, it follows that either $a \in T(p)$ or $Y - T(p) = A \cup B$ sep, where $a \in A$ and $b \in B$. If $a \in T(p)$, $T(p) \cup W$ is a proper subcontinuum of Y containing both a and b; if $a \notin T(p)$, $A \cup T(p) \cup W$ is such a continuum, and in either case we have a contradiction.

COROLLARY 1. *If (Y, X) is a wave from a to b and M is a closed subset of Y with $b \in M$ but $b \notin \operatorname{Int} M$, $\{b\}$ is a component of M.*

LEMMA 7. *If Y is a compact Hausdorff space first countable at a point b, then $Y - \{b\}$ is normal.*

PROOF. Let $\{O_k\}_{k=1}^{\infty}$ be a countable basis of open neighborhoods at b. Then $Y - \{b\} = \bigcup_{k=1}^{\infty} (Y - O_k)$, so that $Y - \{b\}$ is sigma compact and hence Lindelöf. Then $Y - \{b\}$ is paracompact [**10**, p. 174, 6.5] and hence normal [**10**, p. 163, 2.2].

THEOREM 1. *If (Y, X) is a wave from a to b, then γX is an indecomposable continuum.*

PROOF. By Lemma 4, γX is a continuum. Suppose F is a proper subcontinuum of γX which contains an interior point q with respect to γX. Let $p \in \gamma(X) - F$. Let U and V be open sets in βX with $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \mathrm{Cl}(U) \cap (\gamma X - \operatorname{Int} F) = \mathrm{Cl}(V) \cap F = \varnothing$ while $p \in V$ and $q \in U$. This is possible by regularity.

Then $X \cap V$ and $X \cap U$ are open subsets of X and hence of Y since X is open in Y. Let $\langle b_i \rangle_{i=1}^{\infty}$ be a sequence of points in $U \cap X$ converging in Y to b. This is possible since $b \in \mathrm{Cl}_Y(U \cap X)$ and Y is first countable at b.

Then $\{b\}$ is a component of $Y - (V \cap X)$, by Corollary 1, and $\langle b_i \rangle$ is a sequence in $(Y - V) - \{b\}$ converging to b. By Lemma 6 there are two closed sets A_0 and B_0 such that $A_0 \cup B_0 = Y - V$, $A_0 \cap B_0 = \{b\}$, and each of A_0 and B_0 contains a subsequence of the b_i's. Let $A = A_0 \cap X$, $B = B_0 \cap X$. Then A and B are disjoint closed subsets of X, and since X is normal, disjoint closed sets lie in disjoint zero sets, and by Theorem 6.5 III of [**12**], $\mathrm{Cl}_{\beta X}(A) \cap \mathrm{Cl}_{\beta X}(B) = \varnothing$. Now since each of A and B contains infinitely many of the b_i's, it follows that each of $\mathrm{Cl}_{\beta X}(A)$ and $\mathrm{Cl}_{\beta X}(B)$ contains points of $\mathrm{Cl}_{\beta X}(U)\cap\gamma X$, and hence points of F. Thus, since if $x\in\gamma X-\mathrm{Cl}_{\beta X}(A\cup B)$, it follows that $x\in\mathrm{Cl}_{\beta X}(V)$ and hence $x\notin F$, we have $F=(F\cap\mathrm{Cl}_{\beta X}(A))\cup(F\cap\mathrm{Cl}_{\beta X}(B))$ sep, so that F is no continuum.

COROLLARY 2 ([**2**] AND [**3**]). *Let $X=[0, 1)$. Then γX is an indecomposable continuum.*

EXAMPLE 1. Let L denote the long line, consisting of $\omega_1\times[0, 1)$ with the lexicographic order, where ω_1 is the first uncountable ordinal; we take the order topology on L. Then consider $L\times[0, 1]$ with the product topology. Let

$$X = \{((\alpha, t), s)\in L\times[0, 1]: t = 0 \text{ or } t = s\}.$$

Let $Y=X\cup\{b\}$ be the one-point compactification of X. Then Y is irreducible from $((0, 0), 1)$ to b and is connected im Kleinen at b. (Y, X) fails to be a wave from a to b because Y is not first countable at b.

Standard techniques applied to continuous functions from ω_1 to $[0, 1]$ yield the result that $\gamma X\cong[0, 1]$ in this case. Thus, first countability cannot be dispensed with in the hypothesis of Theorem 1. Connectedness im Kleinen also cannot be removed from the hypothesis of Theorem 1; the usual topologist's sin $1/x$ curve, with b taken from the limit arc, yields a decomposable continuum as γX.

LEMMA 8. *If X is a Tychonoff space and Z is any compactification of X with inclusion map $i: X\to Z$, then $\gamma i(\gamma X)=Z-i(X)$.*

REMARK ON PROOF. This is a special case of Theorem 6.12 of [**12**, p. 92].

LEMMA 9. *If X and Y are Tychonoff spaces and $f: X\cong Y$, then γf: $\gamma X\cong\gamma Y$.*

PROOF. By Lemma 8, $\gamma f(\gamma X)=\gamma Y$ and since β is a functor it follows that βf is a homeomorphism since it has inverse $\beta(f^{-1})$. Then γf is a homeomorphism since it is a restriction of one.

LEMMA 10. *Let X be a normal Hausdorff space and A a closed subset of X such that $X-A$ contains a closed but not compact subset of X. Then $\gamma X-\mathrm{Cl}_{\beta X}(A)$ is a nonempty, open subset of γX.*

LEMMA 11. *Suppose X is a Tychonoff space and $f: X\cong X$ is the identity inside some closed subset V of X. Then $\gamma f:\gamma X\cong\gamma X$ is the identity inside $\gamma X\cap\mathrm{Cl}_{\beta X}(V)$.*

DEFINITION. We say a topological space X is *almost homogeneous* if for any $p, q\in X$, and any neighborhood U of q there is a homeomorphism $h: X\cong X$ such that $h(p)\in U$. If, in addition, we may choose h to be the identity on some nonempty open subset of X, we say X is *stably almost homogeneous.*

THEOREM 2. *Let* $X=[0, 1)$; *then* γX *is a stably almost homogeneous indecomposable continuum.*

PROOF. Throughout this proof, Cl denotes $\mathrm{Cl}_{\beta X}$. Let $x, y \in \gamma X$ and let V_0 be any open set in γX containing y. Then $V_0=V_1\cap\gamma X$ for some V_1 open in βX. Then there exists a V_2 open in βX such that $y \in V_2\subseteq \mathrm{Cl}\, V_2\subseteq V_1$ and $x \notin \mathrm{Cl}\, V_2$ unless $x=y$, in which case there is nothing to prove. Let U_0 be open in βX with $x \in U_0$ and $\mathrm{Cl}\, U_0\cap\mathrm{Cl}\, V_2=\varnothing$. Now let $U=U_0\cap X$ and $V=V_2\cap X$. We shall assume, with no loss of generality, that $0<\inf U<\inf V$.

Now, define four sequences $\langle p_n\rangle_{n=1}^{\infty}$, $\langle q_n\rangle_{n=1}^{\infty}$, $\langle r_n\rangle_{n=1}^{\infty}$, and $\langle s_n\rangle_{n=1}^{\infty}$ as follows: $p_1=\inf U$. Whenever p_i has been defined, set $q_i=\sup\{t \in U: [p_i, t]\cap V=\varnothing\}$. When q_i has been defined, set $r_i=\inf\{t \in V: t>q_i\}$. When r_i has been defined, set $s_i=\sup\{t \in V: [r_i, t]\cap U=\varnothing\}$. When s_i has been defined, set $p_{i+1}=\inf\{t \in U: t>s_i\}$. This completes the recursive definition of the four sequences. They have the following properties: $p_i<q_i<r_i<s_i<p_{i+1}$ for each i; the limit in $[0, 1]$ of each sequence is 1, $U\subseteq \bigcup_{i=1}^{\infty} [p_i, q_i]$, and $V\subseteq \bigcup_{i=1}^{\infty} [r_i, s_i]$. We now choose two more sequences $\langle x_i\rangle_{i=1}^{\infty}$ and $\langle y_i\rangle_{i=1}^{\infty}$ so that, for each i, $r_i<x_i<y_i<s_i$ and the closed interval $[x_i, y_i]$ is a subset of V. Finally we choose two more sequences $\langle a_i\rangle_{i=1}^{\infty}$ and $\langle b_i\rangle_{i=1}^{\infty}$ such that $a_1=0$; $0<b_1<p_1$, and for $i>1$ we choose $s_i<a_{i+1}<b_{i+1}<p_{i+1}$. Now define $h:X\cong X$ as follows: For each i,

(1) h is the identity on $[a_i, b_i]$,

(2) h maps the interval $[b_i, p_i]$ linearly onto $[b_i, x_i]$,

(3) h maps $[p_i, q_i]$ linearly onto $[x_i, y_i]$,

(4) h maps $[q_i, a_{i+1}]$ linearly onto $[y_i, a_{i+1}]$.

Then $h(U)\subseteq V$, and hence $\beta h(\mathrm{Cl}(U))\subseteq \mathrm{Cl}(V)$, and since $x \in \mathrm{Cl}(U)$, $\beta h(x) \in \mathrm{Cl}(V)\subseteq \mathrm{Cl}(V_2)\subseteq V_1$, and $\beta h(x)=\gamma h(x) \in V_0$ as desired. Furthermore, γh is the identity inside the set $\gamma X\cap\mathrm{Cl}(\bigcup_{i=1}^{\infty} [a_i, b_i])$, which contains a nonvoid open subset of γX by Lemma 10, setting the closed set $\bigcup_{i=1}^{\infty} [b_i, a_{i+1}]$ equal to A in the lemma.

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