A TWO-COLOR THEOREM FOR ANALYTIC MAPS IN R^n

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ABSTRACT. If $f_1, \dots, f_k$ are real analytic functions on R^n then the connected components of $R^n \setminus [f_1^{-1}(0) \cup \dots \cup f_k^{-1}(0)]$ can be "colored" with two colors so that two components will have different colors whenever their common boundary contains a topological $(n-1)$ -manifold.

If $f_1, \dots, f_k$ are real analytic functions on R^n and if C is a connected component of $R^n \setminus [f_1^{-1}(0) \cup \dots \cup f_k^{-1}(0)]$ then we shall say that C is a *country* of the *analytic map* in R^n determined by $f_1, \dots, f_k$. The *border* of C is its topological boundary ∂C . An *admissible coloring* of a map in R^n is a function whose domain is the set of countries and which assigns different values (colors) to contiguous countries—two countries C, D being *contiguous* if and only if $C \neq D$ and their common border $\partial C \cap \partial D$ contains a topological $(n-1)$ -manifold. Our main result is:

THEOREM. *Every analytic map in R^n can be colored admissibly with two colors.*

It is popularly known that in R^2 some maps require four colors and that five will always suffice [6]. In higher dimensions infinitely many colors may be required for an admissible coloring. For example, consider the map where each country is constructed by connecting a slab perpendicular to the X -axis in the half-space $z > 0$ to a slab perpendicular to the Y -axis in the half-space $z < 0$. More exactly, let a point $(x, y, z, \dots) \in R^n$ lie in the k th country C_k if and only if either (1) $z > 0$ and $k < x < k+1$, or (2) $z = 0$ and $k < x, y < k+1$, or (3) $z < 0$ and $k < y < k+1$. Then each country is contiguous with every other one since $(\partial C_i) \cap (\partial C_j)$ contains the $(n-1)$ -manifold $\{(x, y, z, \dots) \in R^n \mid z = 0, i < x < i+1, j < y < j+1\}$. Furthermore, H. Tietze [7] and A. S. Besicovitch [1] have shown that an infinite number of colors may be required even if the countries of the map are all convex bodies lying in R^3 .

For each $x \in R^n$ let $\mathcal{O}_x$ denote the ring of germs of real analytic functions at x . We denote the germ of the function f at x by f_x and we also interpret

Presented to the Society, January 27, 1973; received by the editors January 11, 1972 and, in revised form, September 27, 1972.

AMS (MOS) subject classifications (1970). Primary 12D10; Secondary 14B05, 05C15.

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" $g_x \in \mathcal{O}_x$ " to mean that g is a real analytic function defined in some open neighborhood of x . The interior of the domain of a function f will be denoted $D[f]$. Each ring $\mathcal{O}_x$ is a unique factorization domain (cf. [4, p. 72]) and when we refer to $uq_1^{e_1} \cdots q_k^{e_k}$ as "the factorization" of $p \in \mathcal{O}_x$ we mean that u (if it appears) is a unit in $\mathcal{O}_x$, the q_i are nonassociate irreducibles in $\mathcal{O}_x$, every integer e_i is positive, and $p = uq_1^{e_1} \cdots q_k^{e_k}$. (Of course k may be 0.) The e_i 's will be called *the exponents of the factorization of p* .

For certain real analytic functions the variety of singular points cannot be very large, as the following proposition shows.

PROPOSITION 1. *Let $f \neq 0$ be a real analytic function on $\mathbf{R}^n$ such that for every $x \in f^{-1}(0)$ all the exponents of the factorization of f_x are 1. Then no topological $(n-1)$ -manifold is contained in the set of singular points of f .*

PROOF. Suppose, on the contrary, that $M \neq \emptyset$ is an $(n-1)$ -manifold on which f and all of its first partial derivatives vanish. Since $f \neq 0$ some partial derivative of f of some order must be nonzero on part of M . Thus we can select g , a partial derivative of f , and $w = (w_1, \dots, w_n) \in M$ so that $g|_M = 0$ but $((\partial/\partial y)g)(w) \neq 0$, where y is one of the x_i 's, say x_n . Now let x denote $(x_1, \dots, x_{n-1})$ so that (x, y) denotes $(x_1, \dots, x_n)$. The Implicit Function Theorem shows that there exists a neighborhood U of w such that the zeros of $g(x, y)$ which lie in U are exactly the graph G of some continuous function $y = \phi(x)$ which is defined in some neighborhood of $(w_1, \dots, w_{n-1}) \in \mathbf{R}^{n-1}$. Thus $M \cap U$ is contained in G since $g|_M = 0$. Moreover $M \cap U$ is a nonempty (it contains w) open subset of the manifold M and so it is locally homeomorphic to $\mathbf{R}^{n-1}$. The mapping $\pi: (x, y) \in G \rightarrow x \in \mathbf{R}^{n-1}$ is a homeomorphism because ϕ is continuous. Thus $V = \pi(M \cap U) \subseteq \mathbf{R}^{n-1}$ is locally homeomorphic to $\mathbf{R}^{n-1}$ and by Invariance of Domain [3] it must be a nonempty open subset of $\mathbf{R}^{n-1}$. For $x \in V$ we know that $f(x, \phi(x)) = (\partial f/\partial y)(x, \phi(x)) = g(x, \phi(x)) = 0$ because $(x, \phi(x)) \in M$.

If we assume that f_w and g_w are relatively prime then by a corollary of the Weierstrass Preparation Theorem (cf. [4, p. 90]) there exist $a_w, b_w \in \mathcal{O}_w$ such that $r_w = a_w f_w + b_w g_w$ is nonzero and the function r depends only on x . Now

$$0 = a(x, \phi(x))f(x, \phi(x)) + b(x, \phi(x))g(x, \phi(x)) = r(x)$$

for every $x \in V' = V \cap D[a(\cdot, \phi(\cdot))] \cap D[b(\cdot, \phi(\cdot))]$. But V' is an open neighborhood of $(w_1, \dots, w_{n-1}) \in \mathbf{R}^{n-1}$, so $r_w = 0$. This contradiction shows that f_w and g_w have an irreducible common divisor. Since w is not a singular point of g , g_w is irreducible, and so g_w must divide f_w , viz. $f_w = g_w h_w$ for some $h_w \in \mathcal{O}_w$. Our hypothesis about the factorization of each f_x when $x \in f^{-1}(0)$ implies that g_w and h_w are relatively prime. Thus there exist $c_w, d_w \in \mathcal{O}_w$ such that $s_w = c_w g_w + d_w h_w$ is nonzero and the function s

depends only on x . Since $g(x, \phi(x))=0$ for $x \in V$, we have $s(x)=d(x, \phi(x))h(x, \phi(x))$ for every x in

$$V'' = V \cap D[c(\cdot, \phi(\cdot))] \cap D[d(\cdot, \phi(\cdot))] \cap D[h(\cdot, \phi(\cdot))].$$

Since $V'' \subseteq \mathbf{R}^{n-1}$ is an open neighborhood of $(w_1, \dots, w_{n-1})$, to contradict $s_w \neq 0$ it suffices to find an open set $W \subseteq \mathbf{R}^{n-1}$ which contains $(w_1, \dots, w_{n-1})$ and on which $h(x, \phi(x))$ vanishes. Let

$$W = \{x \in V \mid (\partial g / \partial y)(x, \phi(x)) \neq 0\} \cap D[h(\cdot, \phi(\cdot))].$$

Since $\partial f / \partial y = (\partial g / \partial y)h + g \partial h / \partial y$, for each $x \in W$ we have $0 = (\partial f / \partial y)(x, \phi(x)) = (\partial g / \partial y)(x, \phi(x))h(x, \phi(x))$. Thus for each $x \in V'' \cap W$ we have $s(x) = d(x, \phi(x))h(x, \phi(x)) = 0$ and so $s_w = 0$.

Now we establish a special case of our theorem.

PROPOSITION 2. *Let $f \neq 0$ be a real analytic function on $\mathbf{R}^n$ such that for every $x \in f^{-1}(0)$ all the exponents of the factorization of f_x are 1. Then $\gamma = \text{sgn}(f)$ induces an admissible coloring of the analytic map determined by f by two colors.*

PROOF. Since γ is constant on each country and since $\gamma(x) = 1$ or -1 when $x \in \mathbf{R}^n \setminus f^{-1}(0)$, it suffices to show that if C and D are two contiguous countries then f is positive in one and negative in the other. Let $M \subset (\partial C \cap \partial D)$ be an $(n-1)$ -manifold and, using the first proposition, pick a regular point $w = (w_1, \dots, w_n) \in M$.

First we use the Implicit Function Theorem to show that w lies in the interior of $\bar{C} \cup \bar{D}$. At least one of the numbers $(\partial f / \partial x_i)(w)$ must be non-zero and we adjust our notation so that this occurs when $i = n$. We also set $y = x_n$ and $x = (x_1, \dots, x_{n-1})$. Then there exists a continuous function ϕ , whose domain V is a connected open neighborhood of $(w_1, \dots, w_{n-1})$ and whose range $\phi(V)$ lies in some interval (a, b) containing w_n such that the zeros of f which lie in $U = V \times (a, b)$ are exactly the graph G of $\phi: V \rightarrow \mathbf{R}$. Since $U_+ = \{(x, y) \in U \mid y > \phi(x)\}$ and $U_- = \{(x, y) \in U \mid y < \phi(x)\}$ are disjoint connected open sets which miss $f^{-1}(0)$ and since $U = U_+ \cup G \cup U_-$ is a neighborhood of w , C must meet, and hence contain, either U_+ or U_- . Likewise D must contain the other. Thus $w \in U \subseteq \text{interior}(\bar{C} \cup \bar{D})$.

Were f positive (or negative) throughout both C and D then f does not change sign in U . Then $f(w) = 0$ is a minimum (or maximum) for f in U and so $(\partial f / \partial x_i)(w) = 0$ for $i = 1, \dots, n$. Since w is not a singular point this is a contradiction and the proposition is established.

Let $\mathcal{O}^c$ denote the sheaf over $\mathbf{C}^n$ of germs of holomorphic functions. Then for each $z \in \mathbf{C}^n$ the ring $\mathcal{O}_z^c$ is a unique factorization domain (cf. [4, p. 72]). We can, and frequently shall, consider $\mathcal{O}_x$ to be a subset of $\mathcal{O}_x^c$ when $x \in \mathbf{R}^n$ because the convergent power series which determines a germ in $\mathcal{O}_x$ also

determines a germ in $\mathcal{O}_x^c$. If $p = g_x \in \mathcal{O}_x^c$ then $\bar{p}$ will denote the germ in $\mathcal{O}_x^c$ of the map $w \rightarrow (g(\bar{w}))^-$. Note that if $x \in \mathbb{R}^n$ and $p = g_x \in \mathcal{O}_x^c$ then $p\bar{p}$ may be viewed as an element of $\mathcal{O}_x$ because $g(y)(g(\bar{y}))^- \in \mathbb{R}$ for every $y \in \mathbb{R}^n \cap D[g]$.

LEMMA. *If p is irreducible in $\mathcal{O}_x$ but reducible in $\mathcal{O}_x^c$ then its factorization in $\mathcal{O}_x^c$ has the form $p = uq\bar{q}$, where u is a unit in $\mathcal{O}_x$ and q and $\bar{q}$ are nonassociate irreducibles in $\mathcal{O}_x^c$.*

PROOF. If $q \in \mathcal{O}_x^c$ is an irreducible which divides p then $\bar{q}$ is an irreducible which divides $\bar{p} = p$. So $q\bar{q} \in \mathcal{O}_x$ and divides p^2 . Since p is reducible in $\mathcal{O}_x^c$, p^2 and $q\bar{q}$ cannot be associates. Thus p and $q\bar{q}$ must be associates in $\mathcal{O}_x$, viz. $p = uq\bar{q}$ for some unit $u \in \mathcal{O}_x$. If q and $\bar{q}$ are associates then there exists a unit $v \in \mathcal{O}_x^c$ such that $p = uvq^2 = ((uv)^{1/2}q)^2$. Let $(uv)^{1/2}q = g_x \in \mathcal{O}_x^c$. Since $p = (g_x)^2$, the function g^2 must be real-valued on the open set $\mathbb{R}^n \cap D[g]$. This implies that one of the analytic functions g, ig must be real-valued throughout $\mathbb{R}^n \cap D[g]$. Thus either g_x or $(ig)_x$ is an irreducible element of $\mathcal{O}_x$ which divides p . But p is irreducible in $\mathcal{O}_x$ and yet $p = (g_x)^2 = -(ig)_x^2$. This contradiction shows that q and $\bar{q}$ are in fact nonassociate.

COROLLARY. *The germs $p, q \in \mathcal{O}_x$ are relatively prime if and only if they are relatively prime in $\mathcal{O}_x^c$.*

Let $\mathcal{O}^*$ denote the sheaf of germs of nowhere vanishing real analytic functions on $\mathbb{R}^n$ and let $\mathcal{M}^*$ denote the sheaf of germs $(f/g)_x$ where $f_x, g_x \in \mathcal{O}_x \setminus \{0\}$ and $x \in \mathbb{R}^n$. Then $\mathcal{O}^*$ may be viewed as a subsheaf of $\mathcal{M}^*$ and the quotient sheaf $\mathcal{D} = \mathcal{M}^*/\mathcal{O}^*$ is the sheaf of real analytic divisors. Letting τ denote the quotient map we obtain the exact sequence

$$(a) \quad 1 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \xrightarrow{\tau} \mathcal{D} \longrightarrow 1.$$

The germ of the set $V \subseteq \mathbb{C}^n$ at $z \in \mathbb{C}^n$ will be denoted V_z , the ideal of that germ, namely $\{f_z \in \mathcal{O}_z^c \mid f_z \text{ vanishes on } V_z\}$, will be denoted $\text{Id}(V_z)$, and for each ideal $I \subseteq \mathcal{O}_z^c$, $\text{Rad}(I) = \{p \in \mathcal{O}_z^c \mid p^n \in I \text{ for some integer } n > 0\}$ is the radical of I (cf. [4]).

PROPOSITION 3. *If f is a real analytic function on $\mathbb{R}^n$ then there exists a real analytic function g on $\mathbb{R}^n$ such that whenever $x \in \mathbb{R}^n$ and $up_1^{e_1} \cdots p_k^{e_k}$ is the factorization of f_x in $\mathcal{O}_x$ then $g_x = vp_1 \cdots p_k$ for some unit $v \in \mathcal{O}_x$.*

PROOF. Let $\delta: \mathbb{R}^n \rightarrow \mathcal{D}$ be defined by $\delta(x) = p_1 \cdots p_k \mathcal{O}_x^* \in \mathcal{D}_x$, where k and the p_i 's depend on x as described above. The first step is to prove that δ is a divisor, i.e. that it is continuous. We shall do this by showing that for each $x \in \mathbb{R}^n$ there exists a real analytic function h defined in some neighborhood U of x such that $\delta(y) = h_y \mathcal{O}_y^*$ for every $y \in U$. In fact, we take h

to be any real analytic function such that $\delta(x) = h_x \mathcal{O}_x^*$, e.g. pick h so that $h_x = p_1 \cdots p_k$.

To see that this choice works we shift our attention from $\mathcal{O}_x$ to $\mathcal{O}_x^c$. If $p_1, \dots, p_j$ are those of the p_i 's which are reducible in $\mathcal{O}_x^c$ then the lemma shows that

$$(b) \quad f_x = w q_1^{e_1} \bar{q}_1^{e_1} \cdots q_j^{e_j} \bar{q}_j^{e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}$$

for some unit $w \in \mathcal{O}_x^c$. Since the corollary shows that $q_1, \bar{q}_1, \dots, p_k$ are nonassociate, (b) must be the unique factorization of f_x in $\mathcal{O}_x^c$. Thus $\text{Rad}(f_x) = (q_1 \bar{q}_1 \cdots p_k) = (p_1 \cdots p_k) = (h_x)$. Applying the Nullstellensatz gives $(h_x) = \text{Rad}(f_x) = \text{Id}([f^{-1}(0)]_x)$, and since $\text{Id}(f^{-1}(0))$ is a coherent sheaf of ideals there exists an open subset W of C^n which contains x such that $(h_z) = \text{Id}([f^{-1}(0)]_z)$ for every $z \in W$ (cf. [4, pp. 138–141]). Let $U = W \cap \mathbf{R}^n$, pick $y \in U$, let $r_1^{d_1} \cdots r_m^{d_m}$ be the factorization of f_y in $\mathcal{O}_y$, and take k to be a real analytic function such that $k_y = r_1 \cdots r_m$. Then $\delta(y) = k_y \mathcal{O}_y^*$, and applying the Nullstellensatz to k as we did to h gives $(k_y) = \text{Id}([f^{-1}(0)]_y)$. But $(h_y) = \text{Id}([f^{-1}(0)]_y)$ also, and so k_y and h_y are associates. Hence $\delta(y) = h_y \mathcal{O}_y^*$, and so this formula must hold for every $y \in U$. Thus δ is a divisor.

The second part of our proof is in the spirit of Cousin's Second Problem. We show that every divisor, and δ in particular, is principal. From (a) we obtain the exact sequence

$$H^0(\mathbf{R}^n, \mathcal{M}^*) \xrightarrow{\tau_*} H^0(\mathbf{R}^n, \mathcal{D}) \longrightarrow H^1(\mathbf{R}^n, \mathcal{O}^*).$$

Since all we wish to know is that τ_* is surjective, we need only prove that $H^1(\mathbf{R}^n, \mathcal{O}^*) = 1$. From the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow \mathbf{Z}_2 \longrightarrow 0$$

we obtain the exact sequence

$$H^1(\mathbf{R}^n, \mathcal{O}) \rightarrow H^1(\mathbf{R}^n, \mathcal{O}^*) \rightarrow H^1(\mathbf{R}^n, \mathbf{Z}_2).$$

Since $H^1(\mathbf{R}^n, \mathbf{Z}_2) = 0$ and $H^1(\mathbf{R}^n, \mathcal{O}) = 0$ (cf. [2, p. 87]), we have $H^1(\mathbf{R}^n, \mathcal{O}^*) = 1$.

The theorem can now be proved quite simply.

PROOF OF THE THEOREM. If $f_1, \dots, f_k$ are the real analytic functions which determine the map then the function $f = f_1 \cdots f_k$ will determine the same map, and so will the function g which can be obtained by applying Proposition 3 to f . Since Proposition 2 can be applied to this g the proof is complete.

REMARKS. (1) No similar two-color or three-color theorem holds in $\mathbf{R}^2$ for C^∞ curves. For example if f is a C^∞ function defined on $\mathbf{R}$ such that $f(x) > 0$ if $|x| < 1$ and $f(x) = 0$ if $|x| \geq 1$ then the curves $y = f(x)$, $y = 0$, and $y = -f(x+1)$ determine a map which requires four colors.

(2) Thinking of countries as vertices and thinking of contiguity as giving edges between vertices will of course give our theorem a graph theoretic interpretation. Some results about coloring the vertices of a graph with two colors can be found in [6].

ACKNOWLEDGEMENT. We are indebted to the referee who outlined for us a method of exploiting local factorization to convert our original theorems about polynomials into theorems about analytic functions. We also thank Professor Hugo Rossi who kindly explained to us about analytic divisors.

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