Pin AND Pin' COBORDISM

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ABSTRACT. The cobordism group $\Omega_*^{\text{Pin}'}$ of smooth manifolds with a Pin structure on the stable normal bundle is computed. The image $\Omega_*^{\text{Pin}'} \rightarrow \mathfrak{N}_*$ is determined, and some generators for Ω_*^{Pin} and $\Omega_*^{\text{Pin}'}$ are given.

1. Notation. All manifolds will be smooth of class C^∞ , and compact. The Lie group $\text{Pin}(n)$ is the double covering of $O(n)$ whose identity component is $\text{Spin}(n)$. (See [3] for details.) Let O , SO , Spin , and Pin denote the usual stable groups. A bundle ξ has a Pin structure if $w_2(\xi) + w_1^2(\xi) = 0$. To see this consider the diagram of fibrations

$$\begin{array}{ccc} B \text{Pin}(1) & \rightarrow & B \text{Pin} \\ \downarrow & & \downarrow \\ BO(1) & \xrightarrow{i} BO & \xrightarrow{f} K(\mathbb{Z}_2, 2). \end{array}$$

We need to find $f^*(i)$. But since the identity component of $B \text{Pin}$ is $B \text{Spin}$, $f^*(i) = w_2 + aw_1^2$ where $a \in \mathbb{Z}_2$. Since $\text{Pin}(1) = \mathbb{Z}_4$, the fibration $B \text{Pin}(1) \rightarrow BO(1)$ is nontrivial, hence $i^*f^*(i) \neq 0$. But $w_2 = 0$ in $BO(1)$, so $a = 1$. (This argument is due to R. E. Stong, and corrects the statement in [6].) A manifold is called a Pin manifold if its stable tangent bundle τ has a Pin structure, and a Pin' manifold if its stable normal bundle ν has a Pin structure. Pin and Pin' do not coincide, since $w_2(\tau) = w_2(\nu) + w_1^2(\nu)$. For example, RP^{4n} is a Pin' manifold, and RP^{4n+2} is a Pin manifold.

In [2] Ω_*^{Pin} was calculated by using the isomorphism $\Omega_*^{\text{Pin}} \approx \Omega_*^{\text{Spin}}(RP^\infty)$. Analogously there is the following long exact sequence, due to Stong, relating Spin and Pin' cobordism, where $\bigwedge$ is the cobordism theory introduced in [4]

$$\cdots \rightarrow \Omega_n^{\text{Spin}} \rightarrow \bigwedge_n \rightarrow \Omega_{n-1}^{\text{Pin}'} \rightarrow \Omega_{n-1}^{\text{Spin}} \rightarrow \cdots$$

2. Algebraic methods. Let M be a closed manifold. Since the stable normal bundle ν of M has a Pin structure iff $w_2(\nu) + w_1^2(\nu) = 0$, the classifying space $B \text{Pin}'$ may be obtained from BO by killing $w_2 + w_1^2$, i.e., $B \text{Pin}'(n)$ is the total space of the fibration over $BO(n)$ induced from the path space

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over $K(\mathbb{Z}_2, 2)$ via the map sending the fundamental class to $w_2 + w_1^2$, and $B \text{ Pin}' = \lim B \text{ Pin}'(n)$. A standard Serre spectral sequence argument shows the following.

LEMMA 2.1. (a) $H^*(B \text{ Pin}'; \mathbb{Z}_2) \approx H^*(BO; \mathbb{Z}_2) / \text{Im}(H^*(K(\mathbb{Z}_2; 2), \mathbb{Z}_2)) \approx \mathbb{Z}_2[w_i, i \neq 2^j + 1, i > 0, j \geq 0]$.

(b) $H^*(B \text{ Pin}'(n), k) \approx H^*(BO(n), k)$ if $k = \mathbb{Z}_p$ or $\mathbb{Q}$, for all n .

Let γ_n be the pullback of the canonical bundle over $BO(n)$ to $B \text{ Pin}'(n)$, and $M \text{ Pin}'(n) = M(\gamma_n)$. Following Thom, there is a cofibration $BO(n) \rightarrow BO(n+1) \rightarrow MO(n+1)$. Lifting this over $B \text{ Pin}'$ gives a cofibration $B \text{ Pin}'(n) \rightarrow B \text{ Pin}'(n+1) \rightarrow M \text{ Pin}'(n+1)$. Then (b) implies $H^*(M \text{ Pin}'(n)) \otimes k \approx H^*(MO(n)) \otimes k$ for $k = \mathbb{Z}_p$ or $\mathbb{Q}$. So $H^*(M \text{ Pin}') \otimes k = 0$, and hence all elements of $\Omega_*^{\text{Pin}'}$ have order a power of 2. For the remainder of this note, all homology and cohomology will be with coefficients $\mathbb{Z}_2$.

Since a Spin bundle has a Pin structure and a Pin' structure, there is a map $\alpha: B \text{ Spin} \rightarrow B \text{ Pin}'$ with $\alpha^*: H^*(B \text{ Pin}') \rightarrow H^*(B \text{ Spin})$ given by $\alpha^*(w_i) = w_i$ if $i \neq 1$ and $\alpha^*(w_1) = 0$. So as a ring but not as an $\mathcal{A}$ module

$$H^*(B \text{ Pin}') \approx H^*(B \text{ Spin}) \otimes \mathbb{Z}_2[w_1]$$

and via the Thom isomorphism

$$H^*(M \text{ Pin}') \approx H^*(M \text{ Spin}) \otimes \mathbb{Z}_2[w_1] \text{ as } \mathbb{Z}_2 \text{ vector spaces.}$$

Also $\Omega_*^{\text{Pin}'}$ is an Ω_*^{Spin} module, since the product of a Spin manifold and a Pin' manifold is a Pin' manifold. Note that $B \text{ Spin}$ is the sphere bundle of the line bundle ξ over $B \text{ Pin}'$ with $w_1(\xi) = w_1$.

We now define the building blocks of $H^*(M \text{ Pin}')$ as an $\mathcal{A}$ module. Let $W, \bar{W}$ be graded vector spaces over $\mathbb{Z}_2$, with a basis for W being $\{x_i\}$, $i \geq 0$, $\deg(x_i) = 4i$ and a basis for $\bar{W}$ being $\{y_i, z_i, u_i, y'_0\}$, $i \geq 0$, with $\deg(y_i) = 4i$, $\deg(z_i) = 4i + 2$, $\deg(u_i) = 4i + 3$, $\deg(y'_0) = 1$. Let M be the quotient of $\mathcal{A} \otimes W$ by the relations $Sq^2 Sq^1(x_0) = 0$, $Sq^2 Sq^3(x_{i-1}) = Sq^1(x_i)$, $i > 0$. Let N be the quotient of $\mathcal{A} \otimes \bar{W}$ by the relations $Sq^1(z_0) = Sq^2(y'_0)$, $Sq^1(z_{i+1}) = Sq^2 Sq^3(z_i) + Sq^2 Sq^2(u_i)$. Note that the submodules of N generated by y_i and u_i are free.

For any finite sequence $J = (j_1, j_2, \dots, j_r)$ of integers, let $n(J) = j_1 + \dots + j_r$. Let V be the graded $\mathbb{Z}_2$ vector space with basis $\{v_J\}$, where J runs over all finite sequences of integers $(j_1, \dots, j_r)$ with $j_i > 1$ and $j_i \leq j_{i+1}$ for all i and $n(J)$ is even. Define $\deg(v_J) = 4n(J)$. Let V' have basis $\{v'_J\}$ with J running through all finite sequences $(j_1, \dots, j_r)$ with $j_i > 1$ and $j_i \leq j_{i+1}$ for all i , and $n(J)$ odd. Define $\deg(v'_J) = 4n(J) - 2$. Then from [1] we have that there is a vector space $\bar{V}$ such that $H^*(M \text{ Spin}) \approx \mathcal{A} / \mathcal{A}(Sq^1, Sq^2) \otimes V \oplus \mathcal{A} / \mathcal{A}(Sq^3) \otimes V' \oplus \mathcal{A} \otimes \bar{V}$, and furthermore $\bar{V}$ has no elements of degree less than 20. Let ξ be an indeterminate of degree 1, and let $V'' = V \otimes \mathbb{Z}_2[\xi]$.

THEOREM 2.1. As an $\mathcal{A}$ module $H^*(M \text{ Pin}')$ is isomorphic to

$$(M \otimes V) \oplus (N \otimes V') \oplus (\mathcal{A} \otimes V'').$$

PROOF. Let $P = M \otimes V \oplus N \otimes V' \oplus \mathcal{A} \otimes V''$. The map $\alpha^*: H^*(M \text{ Pin}') \rightarrow H^*(M \text{ Spin})$ has a $\mathbb{Z}_2$ -module inverse g with $g(w_{i_1} \cdots w_{i_k} U_{\text{Spin}}) = w_{i_1} \cdots w_{i_k} U_{\text{Pin}'}$, where U denotes the Thom class. Let $\alpha_J = 1 \otimes v_J$, $\beta_J = 1 \otimes v'_J$, γ_i be the $\mathcal{A}$ module generators in $H^*(M \text{ Spin})$. Define $f: P \rightarrow H^*(M \text{ Pin}')$ by defining it as follows on the generators, and extending to an $\mathcal{A}$ map.

$$\begin{aligned} f(x_i \otimes v_J) &= w_1^{4i} g(\alpha_J), \\ f(y_i \otimes v'_J) &= w_1^{4i} g(\beta_J), \quad f(y'_0 \otimes v'_J) = w_1 g(\beta_J), \\ f(z_i \otimes v'_J) &= w_1^{4i} S q^2 g(\beta_J), \quad f(u_i \otimes v'_J) = w_1^{4i+3} g(\beta_J), \\ f(1 \otimes \xi^k t_i) &= w_1^k g(\gamma_i), \quad \text{where } t_i \text{ runs through a basis of } \bar{V}. \end{aligned}$$

It is routine to check that the map is well defined on P . For example, if $g(\beta_J) = p_J U$, then $S q^1 p_J = 0 = S q^2 p_J$, so $S q^2 S q^1 (p_J U) = S q^2 (p_J w_1 U) = 0$. Also we have $S q^2 (g(\beta_J)) = S q^1 (w_1 g(\beta_J)) + w_1^2 g(\beta_J)$, and therefore for any integer $j \geq 0$ and for any sequence I , $w_1^j S q^I (g(\alpha_I))$, $w_1^j S q^I (g(\beta_J))$ and $w_1^j S q^I (g(\gamma_i))$ are in the image of f . Since the $\{\alpha_J, \beta_J, \gamma_i\}$ form an $\mathcal{A}$ basis for $H^*(M \text{ Spin})$, and $H^*(M \text{ Pin}') \approx H^*(M \text{ Spin}) \otimes_{\mathbb{Z}_2} [w_1]$ as vector spaces, f must be onto. A counting argument then shows f must be an isomorphism.

The next step is to compute the E_2 term of the Adams spectral sequence. This is $\text{Ext}_{\mathcal{A}}(H^*(M \text{ Pin}'), \mathbb{Z}_2) = \text{Ext}_{\mathcal{A}}(M \otimes V; \mathbb{Z}_2) \oplus \text{Ext}_{\mathcal{A}}(N \otimes V'; \mathbb{Z}_2) \oplus \text{Ext}_{\mathcal{A}}(\mathcal{A} \otimes V''; \mathbb{Z}_2) = \text{Ext}_{\mathcal{A}}(M; \mathbb{Z}_2) \otimes V \oplus \text{Ext}_{\mathcal{A}}(N; \mathbb{Z}_2) \otimes V' \oplus V''$. But $M \approx (M \otimes_{\mathcal{A}} \mathcal{A}_1) \otimes_{\mathcal{A}_1} \mathcal{A}$ and $N \approx (N \otimes_{\mathcal{A}} \mathcal{A}_1) \otimes_{\mathcal{A}_1} \mathcal{A}$ since the only relations

FIGURE 1

FIGURE 2

in M and N come from $\mathcal{A}_1$ ($\mathcal{A}_1$ is the subalgebra of $\mathcal{A}$ generated by 1 , Sq^1 , and Sq^2). Hence we need only compute $\text{Ext}_{\mathcal{A}_1}(M \otimes_{\mathcal{A}} \mathcal{A}_1; Z_2)$ and $\text{Ext}_{\mathcal{A}_1}(N \otimes_{\mathcal{A}} \mathcal{A}_1; Z_2)$. Since $\mathcal{A}_1$ is six dimensional, one can do this as in [4] or via a filtration. Figures 1 and 2 give a picture of the answer in low dimensions.

Here vertical lines denote multiplication by h_0 and diagonal lines multiplication by h_1 , and multiplication by ω corresponds to multiplication by $[\omega] \in \Omega_8^{\text{Spin}}$ (see [2]). $\text{Ext}_{\mathcal{A}}^{s,t}(M, Z_2) = 0$ unless $t-s \equiv 0, 2, 3, 4 \pmod{8}$, and $\text{Ext}_{\mathcal{A}}^{s,t}(N, Z_2) = 0$ for $t-s \not\equiv 0, 2, 3, 4 \pmod{8}$, with $s > 0$.

LEMMA 2.2. *The Adams spectral sequence for $\Omega_*^{\text{Pin}'}$ collapses. Furthermore all extensions, except those given by multiplication by h_0 are trivial.*

PROOF. A theorem of Margolis asserts that if an $\mathcal{A}$ module Q is $H^*(X)$ for some X , and if $Q \approx Q' \oplus F$ where F is a free $\mathcal{A}$ module, then Q is $H^*(X' \vee K)$ where K is an Eilenberg-Mac Lane object, and $H^*(X') \approx Q'$. Hence the free summands in $H^*(M \text{ Pin}')$ cannot give rise to nonzero differentials, nor nontrivial extensions. Since the differentials must commute with multiplication by h_1 , they are all 0. The only possible nontrivial extensions (other than multiplication by h_0) are in dimensions congruent to 2 or 3 mod 8.

But $\Omega_*^{\text{Pin}'}$ is an Ω_*^{Spin} module and the only possible nontrivial extensions in dimensions congruent to 2 mod 8 are $[\omega^k r]$ and $[\omega^l s]$ where $[\omega] \in \Omega_8^{\text{Spin}}$. By induction $[r]$ and $[s]$ have order 2. Since the elements in $\dim 8k+3$ which might possibly support nontrivial extensions are just h_1 times the elements in $8k+2$, their extensions must also be trivial.

$\Omega_n^{\text{Pin}'}$ is just the sum of the contributions from each summand. The

contribution from $M \otimes v_J$ in dimension $4n(J) + 8k + l$ is Z_2 if $l=2, 3$; $Z_{2^{4k+1}}$ if $l=0$ and $Z_{2^{4k+4}}$ if $l=4$, and 0 otherwise. The contribution from $N \otimes v'_J$ in dimension $4n(J) - 2 + 8k + l$ is Z_2 if $l=1, 3, 4, 7$; $Z_2 \oplus Z_2$ if $l=0$, $k>0$, Z_2 if $l=0$, $k=0$, $Z_{2^{4k+2}}$ if $l=2$, $Z_{2^{4k+3}}$ if $l=6$, and 0 otherwise. Thus we have the following table in low dimensions.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\Omega_n^{\text{Pin}'}$	Z_2	0	Z_2	Z_2	Z_2^4	0	0	0	Z_2^5	0	Z_2^2	Z_2^2	$Z_2^8 \oplus Z_2^2$

3. Geometrical considerations. Recall that in [4] a cobordism theory Λ_* was introduced, consisting of pairs (M, ξ) , where ξ is a line bundle over the oriented manifold M , and $w_2(M) = w_1(\xi)^2$. The following theorem is due to Stong.

THEOREM. 3.1. *The sequence*

$$\cdots \longrightarrow \Omega_n^{\text{Spin}} \xrightarrow{\beta} \Lambda_n \xrightarrow{\gamma} \Omega_{n-1}^{\text{Pin}'} \xrightarrow{\delta} \Omega_{n-1}^{\text{Spin}} \longrightarrow \cdots$$

is exact, where $\beta[M] = [M, 1]$, $\gamma[M, \xi] =$ the class of the submanifold of M dual to ξ , and $\delta(N)$ is the class of the oriented double cover of N .

PROOF. Let E_n be the total spaces of the fibration induced from the path fibration by the map $f: BSO(n) \times BO(1) \rightarrow K(Z_2, 2)$ with $f^*(i) = w_2 \otimes 1 + 1 \otimes w_1^2$. Denote the projection $E_n \rightarrow BSO(n) \times BO(1)$ by p . Let γ_n and ξ be the pullbacks to E_n of the canonical bundles over $BSO(n)$ and $BO(1)$. Then as in [4], $\Lambda_k = \lim_n \pi_{n+k}(M(\gamma_n))$. Let X_n be the disc bundles of the pullback of ξ to $D(\gamma_n)$ and let Y_n be the disc bundle of the pullback of ξ to $S(\gamma_n)$. Then $Y_n \subset X_n$, and ∂X_n and ∂Y_n are the corresponding sphere bundles. Hence in the stable range there is an exact sequence

$$\begin{aligned} \cdots \rightarrow \pi_{n+k}(\partial X_n / \partial Y_n) &\rightarrow \pi_{n+k}(X_n / Y_n) \\ &\rightarrow \pi_{n+k}(X_n / Y_n \cup \partial X_n) \rightarrow \pi_{n+k-1}(\partial X_n / \partial Y_n) \rightarrow \cdots \end{aligned}$$

(This makes sense since everything is $(n-1)$ connected.) But $S(\xi) = B \text{Spin}(n)$, and hence $\pi_r(\partial X_n / \partial Y_n) = \pi_r(M \text{Spin}(n))$. Similarly $\pi_r(X/Y) \approx \pi_r(M(\gamma_n))$. Thus we have two of the three groups identified. Now $X_n / Y_n \cup \partial X_n \sim M(\gamma_n \oplus \xi)$. So we need to identify $\lim_n \pi_{n+k}(M(\gamma_n \oplus \xi))$ with $\Omega_{k-1}^{\text{Pin}'}$. But there is a commutative diagram

$$\begin{array}{ccc} E_n & \longrightarrow & B \text{Pin}'(n+1) \\ \downarrow & & \downarrow \\ BSO(n) \times BO(1) & \rightarrow & BO(n+1), \end{array}$$

where the bottom map classifies the product of the canonical bundles, and both horizontal maps are homotopy equivalences up to dimension n . The top map is covered by a bundle isomorphism between $\gamma_n + \xi$ and the canonical bundle over $B \text{Pin}'(n+1)$. This identifies the three groups. The geometric description of the maps is a simple exercise for the reader.

We now need two lemmas which were essentially proved in [4].

LEMMA 3.2. *The kernel of β is the set of those classes in Ω_*^{Spin} which can be written as $[M] \cdot [S^1]^2$, where M is any element of Ω_*^{Spin} and S^1 is the circle with the nonbounding Spin structure.*

LEMMA 3.3. *The element of order 2^{2n+2} in $\bigwedge_{4n+1}$ is $[RP^{4n+1}, \eta]$, where η is the Hopf bundle.*

THEOREM 3.4. (a) *In $\Omega_{4n}^{\text{Pin}'}[RP^{4n}]$ has order 2^{2n+2} if n is odd, and 2^{2n+1} if n is even.*

(b) *In $\Omega_{4n+2}^{\text{Pin}'}[RP^{4n+2}]$ has order 2^{2n+3} if n is even, and 2^{2n+2} if n is odd.*

PROOF. Use 3.1. $\beta[RP^{4n+1}, \eta] = [RP^{4n}]$. Since $[RP^{4n}]$ has filtration 0 in $\Omega_{4n}^{\text{Pin}'}$, and $[RP^{4n+1}, \eta]$ has order 2^{2n+2} , $[RP^{4n}]$ must have order at least 2^{2n+1} . But there is only one such element, and it has the required order. This proves (a). Let $\Delta: \Omega_n^{\text{Pin}'} \rightarrow \Omega_{n-2}^{\text{Pin}'}$ be given by $\Delta(M) = \{\text{the submanifold dual to } \zeta \oplus \zeta \text{ with the inherited Pin structure}\}$ where ζ is the orientation line bundle. Δ is a well-defined homomorphism. Note that $\Delta([RP^{4n}]) = [RP^{4n-2}]$. Similarly define $\Delta': \Omega_n^{\text{Pin}} \rightarrow \Omega_{n-2}^{\text{Pin}'}$. Then $\Delta([RP^{4n+2}]) = [RP^{4n}]$. Now $[RP^{4(n+1)}]$ has order 2^{2n+4} if n is even, and $\Delta' \Delta([RP^{4(n+1)}]) = [RP^{4n}]$, which has order 2^{2n+1} . Thus $[RP^{4n+2}]$ must have order at least 2^{2n+1} and it must have filtration 0. By 2 there is only one such element, and it must have the correct order.

COROLLARY 3.5. *The subgroup of $\Omega_*^{\text{Pin}'}$ generated by elements of order greater than two is just the subgroup generated by the products $[M_J] \times [RP^{4n}]$ where M_J is the generator element in Ω_*^{Spin} described in 1. The same is true for Pin with RP^{4n} replaced by RP^{4n+2} .*

THEOREM 3.6. *The image of $f_*: \Omega_*^{\text{Pin}'} \rightarrow \mathfrak{N}_*$ is the set of cobordism classes all of whose Stiefel-Whitney numbers involving $w_2 + w_1^2$ vanish.*

PROOF. Since the Adams spectral sequence for $\Omega_*^{\text{Pin}'}$ is trivial, and $f^*: H^*(MO) \rightarrow H^*(M \text{Pin}')$ is onto, Theorem 6.2 of [2] implies that

$$\text{Im}(\Omega_n^{\text{Pin}'} \rightarrow \mathfrak{N}_n) = \{g \in \pi_n(MO) \mid g^*(\text{Ker } f^*) = 0 \in H^*(S^n)\}.$$

The argument on p. 468 of [2] shows that this set is precisely those classes whose numbers divisible by $w_1^2 + w_2$ vanish.

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