A VAN KAMPEN THEOREM FOR SEPARABLE ALGEBRAS

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ABSTRACT. If the commutative ring S is the fibre product of R_1 and R_2 over R , and if R_2 maps onto R , then the fundamental group of S is the profinite free product of the fundamental groups of R_1 and R_2 with the fundamental group of R amalgamated.

Van Kampen's theorem in algebraic topology asserts that, under suitable hypotheses, if

$$\begin{array}{ccc} Y & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Z \end{array}$$

is a co-cartesian square of topological spaces, the corresponding diagram

$$\begin{array}{ccc} \Pi_1(Y) & \longrightarrow & \Pi_1(X_1) \\ \downarrow & & \downarrow \\ \Pi_1(X_2) & \longrightarrow & \Pi_1(Z) \end{array}$$

of fundamental groups is also co-cartesian, i.e., that $\Pi_1(Z)$ is the free product of $\Pi_1(X_1)$ and $\Pi_1(X_2)$ with the image of $\Pi_1(Y)$ amalgamated. van Kampen's proof is in [9]; an elementary exposition may be found in [5, Chapter II]. The present paper contains a proof of the analogue of van Kampen's theorem for fundamental groups of commutative rings. (By the fundamental group of a commutative ring we mean the profinite fundamental group in the sense of Grothendieck [8] of the corresponding affine scheme; this is the same as the Galois group of the separable closure of the ring.) As will be seen below, the theorem is a simple consequence of results of Milnor and Bass.

All the rings considered here are commutative with 1, and $\Pi(R)$ denotes the fundamental group of the commutative ring R (for $\Pi(R)$ to be defined R must have no idempotents except zero and one). If G_1 and G_2 are

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profinite groups, $\text{Hom}_c(G_1, G_2)$ denotes the continuous homomorphisms from G_1 to G_2 , and $\mathcal{M}(G_1)$ denotes the category of finite sets on which G_1 acts continuously on the left with morphisms being G_1 -equivariant maps.

DEFINITION 1. Suppose that in (G, G_1, G_2, f_1, f_2) , G, G_1, G_2 are profinite groups and $f_i: G \rightarrow G_i$ are continuous maps. A profinite free product with amalgamation of this five-tuple is a profinite group E and continuous maps $j_i: G_i \rightarrow E$ such that $j_1 f_1 = j_2 f_2$ and for any finite H the map

$$\text{Hom}_c(E, H) \rightarrow \text{Hom}_c(G_1, H) \times_{\text{Hom}_c(G, H)} \text{Hom}_c(G_2, H)$$

given by sending h to (hj_1, hj_2) is a bijection. We denote E by $G_1 \hat{*}_G G_2$ and, for $g_i \in G_i$, $j_1(g_1)j_2(g_2)$ by $g_1 * g_2$.

Profinite free products exist; for lack of an available reference we sketch a proof here.

PROPOSITION 2. Let (G, G_1, G_2, f_1, f_2) be as in Definition 1. Then a profinite free product with amalgamation of this five-tuple exists. The elements of the form $a_1 * b_1 * a_2 * b_2 * \cdots * a_n * b_n$ (where a_i is in G_1 , b_i is in G_2 and n ranges over the positive integers) are dense in the free product.

PROOF. We denote the ordinary free product with amalgamation of (G, G_1, G_2, f_1, f_2) by $G_1 *_G G_2$ (see [6, Chapter 17] for the theory of such products). Let $h_i: G_i \rightarrow H$ be a pair of continuous homomorphisms with range the finite group H such that $h_1 f_1 = h_2 f_2$, and let $N(h_1, h_2)$ be the kernel of the homomorphism $G_1 *_G G_2 \rightarrow H$ induced by the h_i . Since we have (in the obvious notation) $N(h_1, h_2) \cap N(k_1, k_2) = N(h_1 \times k_1, h_2 \times k_2)$ the set $\mathcal{N}$ of all such subgroups is a basis at the identity for a topology on $G_1 *_G G_2$ and in this topology the homomorphisms $G_i \rightarrow G_1 *_G G_2$ are continuous. $\mathcal{N}$ is directed by inverse inclusion and we let $G_1 \hat{*}_G G_2$ be the inverse limit of $G_1 *_G G_2 / N$ as N ranges over $\mathcal{N}$ and $j_i: G_i \rightarrow G_1 \hat{*}_G G_2$ the composite of $G_i \rightarrow G_1 *_G G_2$ followed by the projection to the inverse limit. We omit the easy verification that this is indeed a profinite free product with amalgamation. The second assertion of the theorem follows from the fact that the projection $G_1 *_G G_2 \rightarrow G_1 *_G G_2 / N$ is onto at each stage of the inverse system.

Let C, C_1, C_2 be categories and $F_i: C_i \rightarrow C$ be functors. Following Bass [1, 3.1, p. 358] we define the fibre product category $C_1 \times_C C_2$ to be the category whose objects are triples (X_1, a, X_2) where X_i is an object of C_i and $a: F_1 X_1 \rightarrow F_2 X_2$ is an isomorphism, and whose morphisms are pairs $(h_1, h_2): (X_1, a, X_2) \rightarrow (Y_1, b, Y_2)$ where $f_i: X_i \rightarrow Y_i$ is a morphism in C_i and $F_2 h_2 a = b F_1 h_1$.

PROPOSITION 3. *Let $(G, G_1, G_2, f_1 f_2)$ be as in Definition 1. Then $\mathcal{M}(G_1 \hat{*}_G G_2)$ is equivalent to $\mathcal{M}(G_1) \times_{\mathcal{M}(G)} \mathcal{M}(G_2)$. (Here the functor from $\mathcal{M}(G_i)$ to $\mathcal{M}(G)$ is "restriction of operators".)*

PROOF. An object of the fibre product category is a triple (X_1, a, X_2) where X_i is a G_i -set, $a: X_1 \rightarrow X_2$ is a bijection and, for $g \in G$ and $x \in X_1$, $a(f_1(g)x) = f_2(g)(ax)$. Let $A(X_i)$ be the symmetric group in the set X_i and let $g_i: G_i \rightarrow A(X_i)$ give the G_i -structure on X_i . Define $g'_1: G_1 \rightarrow A(X_2)$ by $g'_1(g) = ag_1(g)a^{-1}$ for $g \in G_1$. Then $g'_1 f_1 = g_2 f_2$ and since there is an induced continuous map $G_1 *_G G_2 \rightarrow A(X_2)$ which makes X_2 into a $G_1 \hat{*}_G G_2$ -set which we call $X(X_1, a, X_2)$. If $c_i \in G_1$, $d_i \in G_2$, $i = 1, 2, \dots, n$ and $x \in X(X_1, a, X_2)$, then $(c_1 * d_1 * \dots * c_n * d_n)x = ac_1 a^{-1} d_1 \dots ac_n a^{-1} d_n x$. Let $(h_1, h_2): (X_1, a, X_2) \rightarrow (Y_1, b, Y_2)$ be a morphism in the fibre product category so $h_i: X_i \rightarrow Y_i$ is a G_i map such that, as functions, $h_2 a = b h_1$ (note then that given a, b and h_2 , h_1 (if it exists) is unique since $h_1 = b^{-1} h_2 a$). Let $X(h_1, h_2): X(X_1, a, X_2) \rightarrow X(Y_1, a, Y_2)$ be the function $h_2: X_2 \rightarrow Y_2$. We claim that $X(h_1, h_2)$ is a morphism of $G_1 \hat{*}_G G_2$ -sets. Clearly it suffices to check this in a dense subset of the free product, i.e., by Proposition 2, for the elements $c_1 * d_1 * \dots * c_n * d_n$ with $c_i \in G_1$ and $d_i \in G_2$. To simplify notation we take $n = 1$. Then, for x in X_2 , c in G_1 and d in G_2 ,

$$X(h_1, h_2)(c * d)(x) = h_2 a c a^{-1} d x = b c b^{-1} d h_2 x = (c * d) X(h_1, h_2)(x).$$

We note that $X(\cdot)$ is a functor from the fibre product category to $\mathcal{M}(G_1 \hat{*}_G G_2)$ which we claim is an equivalence. The functor is clearly faithful (see the note above). To see it is full, let $h: X(X_1, a, X_2) \rightarrow X(Y_1, b, Y_2)$ be a $G_1 \hat{*}_G G_2$ -map. Let $h_2: X_2 \rightarrow Y_2$ be h (as a function) and let $h_1 = b^{-1} h_2 a$. Let $c \in G_1$ and $d \in G_2$. Then it is easy to check from the equations $h(g_1 * 1)(ax) = (g_1 * 1)h(ax)$ (for $x \in X_1$) and $h(1 * g_2)(x) = (1 * g_2)h(x)$ (for $x \in X_2$) that h_i is G_i -equivalent. By definition $h_2 a = b h_1$ so (h_1, h_2) is a morphism in the fibre product category and $X(h_1, h_2) = h$. Thus $X(\cdot)$ is full. Finally, to complete the proof that $X(\cdot)$ is an equivalence, we verify that every isomorphism class in $\mathcal{M}(G_1 \hat{*}_G G_2)$ contains an image of $X(\cdot)$. Let X be any object of $\mathcal{M}(G_1 \hat{*}_G G_2)$. Make X a G_i -set via $g_i x = j_i(g_i)x$ for $g_i \in G_i$, $x \in X$, and call this G_i -set X_i . Let $a: X_1 \rightarrow X_2$ be the identity. Then (X_1, a, X_2) belongs to the fibre product category and the identity function provides a $G_1 \hat{*}_G G_2$ -isomorphism $X(X_1, a, X_2) \rightarrow X$.

We now turn our attention to separable algebras. We begin by recalling Milnor's theorem ([7, 2.1–2.3, p. 20] and [1, Theorem 5.1, p. 497]): if

$$\begin{array}{ccc} A & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & B \end{array}$$

is a cartesian square of commutative rings with $A_2 \rightarrow B$ surjective, then $P(A)$ is equivalent to $P(A_1) \times_{P(B)} P(A_2)$ (where $P(R)$ is the category of finitely generated projective modules over the ring R); the equivalence sends an object M of $P(A)$ to $(A_1 \otimes_A M, a, A_2 \otimes_A M)$ where

$$a: B \otimes_{A_1} (A_1 \otimes_A M) \rightarrow B \otimes_{A_2} (A_2 \otimes_A M)$$

is the canonical isomorphism and the inverse functor to the equivalence sends (M_1, a, M_2) to $\{(x, y) \text{ in } M_1 \times M_2 : a(1 \otimes x) = 1 \otimes y\}$.

As Bass remarks [1, (5.2), p. 481], Milnor's theorem also applies to various categories of projective modules with structure; one need only check that the equivalence and its inverse functor take values in the subcategory of the modules carrying the structure in question. We apply this to the category of commutative separable algebras whose underlying module is finitely generated and projective (over the ring R this category will be denoted $S(R)$).

PROPOSITION 4. *Under the hypotheses of Milnor's theorem, $S(A)$ is equivalent to $S(A_1) \times_{S(B)} S(A_2)$.*

PROOF. If T is an object in $S(A)$, then $A_i \otimes_A T$ is an object in $S(A_i)$, and it follows that the equivalence of Milnor's theorem takes values in the fibre product category. Now let (T_1, a, T_2) be an object of the fibre product category. We need to verify that $M(T_1, a, T_2) = M$ (which is clearly an A -algebra and by Milnor's theorem a finitely generated projective A -module) is a separable A -algebra. Since it is finitely generated and projective as an A -module, M will be separable if for each maximal ideal p of A , M/pM is a separable A/pA -algebra [4, Theorem 7.1(c), p. 72]. But by [1, Proposition 5.11, p. 486], there is a prime ideal q in either A_1 or A_2 (say A_1) with $q \cap A = p$. By [7, Theorem 2.3, p. 20], $A_1 \otimes_A M = T_1$, so $M/pM \otimes_{A/pA} A_1/qA_1 = T_1/qT_1$ is A_1/qA_1 -separable. Since A_1/qA_1 contains A/pA as A/pA -summand, M/pM is A/pA -separable [4, Corollary 1.10, p. 45].

Childs gives the analogue of Proposition 4 for Azumaya algebras in [3, §IV]. We now proceed to the main result of this paper.

THEOREM 5. *Let*

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & B \end{array}$$

be a cartesian square of commutative rings with no nontrivial idempotents with $A_1 \rightarrow B$ surjective. Then $\Pi(A) = \Pi(A_1) \hat{\star}_{\Pi(B)} \Pi(A_2)$.

PROOF. We begin by remarking that the dual of a fibre product category is the fibre product of the duals. The categories $\mathcal{M}(\Pi(A))$, $\mathcal{M}(\Pi(A_1))$, $\mathcal{M}(\Pi(A_2))$ and $\mathcal{M}(\Pi(B))$ are the dual categories to $S(A)$, $S(A_1)$, $S(A_2)$ and $S(B)$ [8, Chapter IV]. Thus by Proposition 4 and the above remark, $\mathcal{M}(\Pi(A))$ is equivalent to $\mathcal{M}(\pi(A_1)) \times_{\mathcal{M}(\Pi(B))} \mathcal{M}(\Pi(A_2))$. By Proposition 3 this latter category is equivalent to $(\Pi(A_1) \hat{*}_{\Pi(B)} \Pi(A_2))$. By [8, 4.4.1.11, p. 87] it follows that $\Pi(A)$ and $\Pi(A_1) \hat{*}_{\Pi(B)} \Pi(A_2)$ are isomorphic since the categories of finite sets on which they act continuously are equivalent.

A list of common situations in which the hypotheses of Theorem 5 are satisfied is found in [1, p. 483]. We consider some representative examples here.

EXAMPLE A. Let G be a cyclic group of prime order p and Z a primitive p th-root of unity. Then [7, p. 29]

$$\begin{array}{ccc} ZG & \longrightarrow & Z[z] \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z/pZ \end{array}$$

where the left vertical map is augmentation, the right vertical sends z to 1, the top map sends a generator of G to z and the bottom map is the canonical one is cartesian. Theorem 5 applies and we conclude, using the fact that $\Pi(Z)$ is trivial, that $\Pi(Z[z]) \rightarrow \Pi(ZG)$ is surjective with kernel the image of $\Pi(Z/pZ)$.

EXAMPLE B. Let the compact connected Hausdorff space X be the union of the two connected closed subsets F_1 and F_2 and suppose $Y = F_1 \cap F_2$ is also connected. Let $C(\cdot)$ denote the ring of continuous complex valued functions on $(\cdot)$. Then

$$\begin{array}{ccc} C(X) & \longrightarrow & C(F_2) \\ \downarrow & & \downarrow \\ C(F_1) & \longrightarrow & C(Y) \end{array}$$

is cartesian. By a result of Childs [2, Theorem 2, p. 30], $\Pi(C(X))$ is the profinite completion of $\Pi_1(X)$, and similarly for F_1 , F_2 and Y . Thus from Theorem 5 we recover a topological van Kampen theorem, at least up to profinite completion.

EXAMPLE C. Let R be the real numbers, C the complexes, and A the ring of polynomials f over R in the variable X such that $f(\sqrt{-1})$ is real.

Then

$$\begin{array}{ccc} A & \longrightarrow & R[X] \\ \downarrow & & \downarrow \\ R & \longrightarrow & C \end{array}$$

where the horizontal maps are the inclusions, the left map and the right evaluation at $\sqrt{-1}$, satisfies the hypotheses of Theorem 5. Of course, $\Pi(C)$ is trivial and $\Pi(R)=\mathbb{Z}/2\mathbb{Z}$. Also $\Pi(R[X])=\mathbb{Z}/2\mathbb{Z}$ (since $C[X]$ is separably closed and Galois over $R[X]$ with group $\mathbb{Z}/2\mathbb{Z}$) and hence it follows that $\Pi(A)=\mathbb{Z}/2\mathbb{Z} \hat{*} \mathbb{Z}/2\mathbb{Z}$ (we note that this latter group is non-abelian and infinite since it has every dihedral group as a homomorphic image).

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