Chebyshev Estimates for Beurling Generalized Prime Numbers

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ABSTRACT. We consider a Beurling generalized number system for which the counting function of the integers satisfies

$$N(x) = cx + O(x \log^{-\gamma} x)$$

for some positive c and γ . Beurling showed that the prime number theorem must hold if $\gamma > \frac{3}{2}$, but it can fail to hold if $\gamma \leq \frac{3}{2}$. It was shown by Hall in the preceding article that the Chebyshev prime counting estimates

$$0 < \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x}; \quad \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} < \infty$$

can fail if $\gamma < 1$. Here we shall prove that these estimates hold for any system satisfying $\gamma > 1$. The proof uses a convolution approximate inverse of the measure dN .

Chebyshev established the correct order of magnitude of the weighted prime counting function ψ by exhibiting numbers $a > 0$ and $A < \infty$ such that

$$(1) \quad \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x} \geq a, \quad \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} \leq A.$$

The prime number theorem (P.N.T.) asserts that $a=A=1$. Here we shall study Chebyshev estimates for Beurling generalized prime numbers.

The first paragraph of the preceding paper by Hall mentions the basic facts about Beurling generalized numbers. For a more detailed discussion, see e.g. [1] or [2]. The generalized psi function is defined on $[1, \infty)$ by setting

$$\psi(x) = \sum_{p_i^{\alpha} \leq x} \log p_i$$

where $p_1, p_2, p_3, \dots$ is the sequence P of generalized primes and α ranges over positive integers. All other notation is as in Hall's paper.

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Beurling [2] proved that if

$$(2) \quad N(x) = cx + O(x \log^{-\gamma} x)$$

for some $c > 0$ and $\gamma > \frac{3}{2}$, then $\psi(x) \sim x$, i.e., the P.N.T. holds for P . Also, he showed by example that the P.N.T. could fail for $\gamma \leq \frac{3}{2}$. In the preceding article and in [5] Hall proves that the Chebyshev estimates can fail for $\gamma < 1$. In the present paper we study the remaining range $1 < \gamma \leq \frac{3}{2}$ and prove the following

THEOREM. *If P is a generalized prime number system for which N_P satisfies (2) with $c > 0$ and $\gamma > 1$, then there exist numbers $a > 0$ and $A < \infty$ for which (1) holds.*

In [1] Bateman and the author had conjectured that the upper Chebyshev estimate holds when $\gamma > 1$. A similar conjecture was recently mentioned to the author by Professor J. H. Chalk. This estimate has applications to tauberian theorems of T. M. K. Davison [3] and S. L. Segal [6].

We begin by proving the upper estimate; the lower estimate then follows easily. The method of Chebyshev, which exploits the relation

$$\psi(2x) - \psi(x) + \sum_{j=1}^{\infty} \left\{ \psi\left(\frac{x}{j+1/2}\right) - \psi\left(\frac{x}{j+1}\right) \right\} = 2x \log 2 + O(\log x),$$

appears to be of no use in the present situation.

Let $F: [1, \infty) \rightarrow \mathbf{R}$ be of bounded variation on each interval $[1, x]$. There is an associated Borel-Stieltjes measure on $[1, \infty)$ which we denote by dF . We define an operator L on measures by setting

$$\int_E L dF = \int_{t \in E} \log t dF(t).$$

Also, we employ multiplicative convolution of measures, denoted by $*$. The starting point of our proof is the Chebyshev formula $\sum_{d|n} \Lambda(d) = \log n$ which, in the form

$$(3) \quad d\psi * dN = L dN$$

is valid for Beurling generalized numbers (see [4]). We convolve each side of (3) by $(\delta - \varepsilon t^{-\varepsilon} dt) * (\log et)^{-\gamma} dt$, where δ denotes Dirac measure at 1, dt is Borel-Lebesgue measure on $[1, \infty)$ and ε is a positive number presently to be specified. We shall show that the new factors act like an approximate convolution inverse of dN . Let

$$\nu_{\varepsilon}(x) := \int_1^x dN * (\delta - \varepsilon t^{-\varepsilon} dt) * (\log et)^{-\gamma} dt.$$

LEMMA 1. Assume that N satisfies (2) with $c > 0$ and $\gamma > 1$. There exists a positive number ε , depending on N , such that $v_\varepsilon(x) \geq 0$ for all $x \geq 1$ and $v_\varepsilon(x) \rightarrow \infty$ as $x \rightarrow \infty$.

PROOF. We write

$$N(x) = cx + \theta Kx(\log ex)^{-\gamma}$$

where we may suppose that $1 < \gamma < 2$, $K > 0$ and $|\theta| \leq 1$. Let

$$\varepsilon = \exp\{-16eK/c(\gamma - 1)\}.$$

If $1 \leq x \leq 1/\varepsilon$, then

$$\int_1^x dN * (\delta - \varepsilon t^{-\varepsilon} dt) \geq N(x) - \varepsilon N(x)(x - 1) \geq 0.$$

The lemma is certainly true for $1 \leq x \leq 1/\varepsilon$ since the third convolution factor is everywhere nonnegative.

If $x > 1/\varepsilon$, we utilize all the convolution factors. We write

$$\begin{aligned} v_\varepsilon(x) &= c \int_1^x (\delta + dt) * (\delta - \varepsilon t^{-\varepsilon} dt) * (\log et)^{-\gamma} dt \\ &\quad + \int_1^x (dN - c\delta - c dt) * (\delta - \varepsilon t^{-\varepsilon} dt) * (\log et)^{-\gamma} dt \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

We evaluate I_1 by iterated integration, first combining the first and second factors and then changing the integration variable. We find that

$$I_1 = cx \int_1^x t^{-\varepsilon-1} (\log ex/t)^{-\gamma} dt.$$

To estimate $|I_2|$ let

$$d\alpha(t) = \delta + \varepsilon t^{-\varepsilon} dt \quad \text{and} \quad d\beta(t) = (\log et)^{-\gamma} dt.$$

$$|I_2| \leq \int_1^x Kxt^{-1} (\log ex/t)^{-\gamma} (d\alpha * d\beta)(t).$$

We again iterate the integration, obtaining first

$$\begin{aligned} \int_1^x Kxt^{-1} (\log ex/t)^{-\gamma} d\beta(t) &\leq \int_1^{\sqrt{x}} Kx(\log e\sqrt{x})^{-\gamma} t^{-1} (\log et)^{-\gamma} dt \\ &\quad + \int_{\sqrt{x}}^x Kx(\log ex/t)^{-\gamma} t^{-1} (\log e\sqrt{x})^{-\gamma} dt \\ &\leq 8Kx(\log ex)^{-\gamma}/(\gamma - 1). \end{aligned}$$

Then

$$\begin{aligned} |I_2| &\leq \frac{8K}{\gamma-1} \int_1^x \frac{x}{t} \left(\log \frac{ex}{t} \right)^{-\gamma} (\delta + \varepsilon t^{-\varepsilon} dt) \\ &= \frac{8Kx}{\gamma-1} (\log ex)^{-\gamma} + \frac{8K\varepsilon x}{\gamma-1} \int_1^x t^{-\varepsilon-1} \left(\log \frac{ex}{t} \right)^{-\gamma} dt. \end{aligned}$$

Now $8K\varepsilon/(\gamma-1) < c/3$ by the choice of ε . Next, we show that for $x > 1/\varepsilon$

$$\frac{8Kx}{\gamma-1} (\log ex)^{-\gamma} < \frac{1}{2} cx \int_1^x t^{-\varepsilon-1} (\log ex/t)^{-\gamma} dt.$$

Indeed, we have

$$\begin{aligned} \frac{1}{2} cx \int_1^x t^{-\varepsilon-1} (\log ex/t)^{-\gamma} dt &> \frac{1}{2} cx (\log ex)^{-\gamma} \int_1^{1/\varepsilon} t^{-\varepsilon-1} dt \\ &> \frac{c}{2} x \varepsilon^\varepsilon \left(\log \frac{1}{\varepsilon} \right) (\log ex)^{-\gamma} > \frac{8Kx}{\gamma-1} (\log ex)^{-\gamma} \end{aligned}$$

since $\varepsilon^\varepsilon \geq \exp(-1/\varepsilon)$ for $0 < \varepsilon < 1$. Now

$$v_\varepsilon(x) \geq I_1 - |I_2| \geq \frac{1}{8} cx \int_1^x t^{-\varepsilon-1} (\log ex/t)^{-\gamma} dt.$$

The last term is clearly positive and tends to infinity with x . $\square$

LEMMA 2.

$$\int_1^x L dN * (\delta - \varepsilon t^{-\varepsilon} dt) * (\log et)^{-\gamma} dt = O(x).$$

PROOF. We apply iterated integration. First

$$\int_1^x L dN * (\delta - \varepsilon t^{-\varepsilon} dt) = cx/\varepsilon + o(x) = O(x).$$

Next,

$$\int_1^x O\left(\frac{x}{t}\right) (\log et)^{-\gamma} dt = O(x). \quad \square$$

PROOF OF THE UPPER ESTIMATE. Formula (3) yields

$$\begin{aligned} \int_1^x dN * (\delta - \varepsilon t^{-\varepsilon} dt) * \frac{dt}{(\log et)^\gamma} * d\psi \\ = \int_1^x L dN * (\delta - \varepsilon t^{-\varepsilon} dt) * \frac{dt}{(\log et)^\gamma}. \end{aligned}$$

The right-hand side of this equation was just shown to be $O(x)$. The left side is representable as $\int_1^x v_e(x/t) d\psi(t)$. We have shown that $v_e(u) \geq 0$ for all $u \geq 1$ and that there exists a number B for which $v_e(u) \geq 1$ for all $u \geq B$. Also ψ is an increasing function. Thus we have

$$\psi(x/B) \leq \int_1^x v_e(x/t) d\psi(t) = O(x),$$

or $\limsup_{x \rightarrow \infty} \psi(x)/x < \infty$.

PROOF OF THE LOWER ESTIMATE. Easy calculations show that

$$\begin{aligned} \rho(x) &:= \int_1^x dN * \left(\delta - \frac{dt}{t} \right) = O\{x(\log ex)^{-\gamma}\}, \\ \int_1^x L dN * \left(\delta - \frac{dt}{t} \right) &= cx + O\{x(\log ex)^{1-\gamma}\}. \end{aligned}$$

We convolve both sides of (3) by $\delta - t^{-1} dt$ and use the last estimates. We have

$$\begin{aligned} cx + O(x \log^{1-\gamma} x) &= \int_1^x \rho(x/t) d\psi(t) \\ &\leq K \int_1^x \frac{x}{t} \left(\log \frac{ex}{t} \right)^{-\gamma} d\psi(t). \end{aligned}$$

Integration by parts and omission of a negative term yield

$$cx + o(x) \leq K \left\{ \psi(x) + \int_1^x \psi \left(\frac{x}{u} \right) (\log eu)^{-\gamma} du \right\}.$$

By the upper estimate for ψ there exists a number M such that $\psi(x) \leq Mx$ for all positive x . Now choose $B \geq 1$ so large that

$$KM \int_B^\infty \frac{x}{u} (\log eu)^{-\gamma} du < \frac{1}{3} cx.$$

If x is large enough so that $o(x) > -(1/3)cx$, then we have

$$\frac{1}{3} cx < K\psi(x) + K\psi(x) \int_1^B (\log eu)^{-\gamma} du \leq KB\psi(x).$$

Consequently,

$$\liminf_{x \rightarrow \infty} \psi(x)/x \geq \frac{c}{3KB} > 0. \quad \square$$

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