THE BRAUER GROUP OF AN AMITSUR FIELD

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ABSTRACT. In this paper we compute the entire Brauer groups of Amitsur fields (i.e. certain fields of formal power series K_m described in §1); these are interesting generalizations of the Brauer groups over the fields of local class field theory.

1. Power series fields. Throughout this paper, we assume K is a field of characteristic zero. Denote by $K\{t\}$ the field of all formal power series $p\{t\} = \sum_{v \geq r} a_v t^v$, $a_v \in K$, $r > -\infty$. Let $t_1, \dots, t_m$ be a finite set of indeterminates over K . Define successively, $K_0 = K$, $K_1 = K_0\{t_1\}$, $K_j = K_{j-1}\{t_j\} = K\{t_1\}\{t_2\} \cdots \{t_j\}$. Thus the general element of K_j has the form:

$$p\{t\} = p\{t_1, \dots, t_j\} = \sum_{v_j > r_j} \sum_{v_{j-1} \geq r_{j-1}} \cdots \sum_{v_1 \geq r_1} \alpha_{v_1 v_2 \cdots v_j} t_1^{v_1} t_2^{v_2} \cdots t_j^{v_j}$$

where $r_i = r_i(v_{i+1}, \dots, v_j) > -\infty$ for $i = 1, \dots, j-1$, $r_j > -\infty$. Clearly, each K_j is a field.

If K is an algebraically closed field, the power series field K_m is called an Amitsur field of m -indeterminates (or simply an Amitsur field). Throughout this paper we assume K is an algebraically closed field and K_m is an Amitsur field. Let

$$(1) \quad p\{t\} = \sum_{v_m \geq r_m} \sum_{v_{m-1} \geq r_{m-1}} \cdots \sum_{v_1 \geq r_1} \alpha_{v_1 v_2 \cdots v_m} t_1^{v_1} t_2^{v_2} \cdots t_m^{v_m} \in K_m.$$

The smallest exponent v_m of t_m occurring in any nonzero terms of (1) is called the order of $p\{t\}$ and denoted by $o_m(p\{t\})$. Define a valuation $|\cdot|_m$ on K_m as follows: $|p\{t\}|_m = e^{-o_m(p\{t\})}$ where e is any real number greater than 1. Then $|\cdot|_m$ is a discrete, nonarchimedean valuation with the ring of integers $\mathfrak{O}_m = K_{m-1}[[t_m]]$ (the ring of formal powers series over K_{m-1}); maximal ideal $\mathfrak{P}_m = \{p\{t\} \in \mathfrak{O}_m \mid o_m(p\{t\}) \geq 1\}$ and the residue class field $\bar{K}_m = \mathfrak{O}_m / \mathfrak{P}_m \cong K_{m-1}$. It is easy to see that K_0 has trivial valuation $|\cdot|_0$; K_1 is a quasi-finite field (i.e. a field which has no inseparable extension and, for each n , it has exactly one extension of degree n in its algebraic closure)

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and K_2 is a regular local field (i.e. a field which is complete under a discrete, nonarchimedean valuation with a residue class field which is quasi-finite).

2. The character group of K_m . Denote the algebraic closure of the field L by L^c and the Galois group of L^c over L by $G(L)$.

Let $\hat{Z}$ be the Artin group which is the completion of Z with the topology of subgroups of finite index. Then $\hat{Z}$ is a compact, totally disconnected group which is identical with the projective limit of Z/nZ and is canonically isomorphic to the product $\prod Z_p$ over all primes p , Z_p the ring of p -adic integers. If L is a quasi-finite field, then it follows easily that $G(L) = \hat{Z}$ [5, §2, p. 198].

By a character over the field L , we mean a homomorphism of $G(L)$ into Q/Z (=the rationals mod 1) which is continuous (Krull topology). Let $\chi(L)$ be the group of these characters over L . If L is a quasi-finite field, then it follows from [5, Proposition 1 and Corollary, p. 197] that $\chi(L) \cong Q/Z$. In the following, we use induction on m and the above result to find $\chi(K_m)$.

2.1 LEMMA. *Let $p\{t\} = 1 + \sum_{i=1}^{\infty} \alpha_i t_m^i \in K_m$ where $\alpha_i \in K_{m-1}$. Then $p\{t\} \in K_m^n = \{\alpha^n \mid \alpha \in K_m\}$ for every $n \in Z^+$.*

PROOF. Let $q\{t\} = 1 + \sum_{i=1}^{\infty} \beta_i t_m^i \in K_m$ where $\beta_i \in K_{m-1}$; then $q\{t\}^n = 1 + \sum_{i=1}^{\infty} r_i t_m^i$ where $r_i \in K_{m-1}$ and $r_i = r'_i(\beta_1, \beta_2, \dots, \beta_{i-1}) + n\beta_i$ is a function of $\beta_1, \beta_2, \dots, \beta_i$. Hence there exist β_i , $i=1, 2, \dots$, such that $r_i = \alpha_i$ for all i . Therefore, $p\{t\} = q\{t\}^n \in K_m^n$.

2.2 PROPOSITION. *Every finite extension of K_m is of the form $K_m(A, T)$ where A is algebraic over K_{m-1} and $T^n = \alpha t_m$ for some $n \in Z^+$ and $\alpha \in K_{m-1}$.*

PROOF. $K_m = K_{m-1}\{t_m\}$ is complete with respect to the valuation $|\cdot|_m$. The subfield K_{m-1} is a set of representatives for the residue class field and since the characteristic is 0, it follows easily from Hensel's lemma that if L/K_m is finite algebraic, then its maximal unramified subfield is of the form $K_m(A)$ with A algebraic over K_{m-1} . Then $L/K_m(A)$ is pure ramified, hence generated by an element θ with $|\theta^n|_m = |t_m|_m$ where n is the ramification number; i.e. $\theta^n = \alpha(1 + \pi)t_m$ with $\alpha \in K_{m-1}$ and π an element of L with $|\pi| < 1$, in the unique extension of $|\cdot|_m$ to L .

By Lemma 2.1, which applies here because L is a field of formal power series in θ , there is a $\gamma \in L$ with $\gamma^n = 1 + \pi$. Set $T = \theta\gamma^{-1}$; then $L = K_m(A, T)$. Q.E.D.

By induction on m and Proposition 2.2 we get:

2.3 PROPOSITION. *K_m^c is generated by adjoining roots of the equations $x^n = t_i$, where n is any positive integer and $i=1, \dots, m$.*

2.4 PROPOSITION. $G(K_m^c/K_m) \cong \bigoplus_{i=1}^m \hat{Z}$.

PROOF. Let σ be any automorphism of K_m^c/K_m ; it induces an automorphism σ_0 on K_{m-1}^c/K_{m-1} and it is easy to see that we can extend σ_0 to an automorphism σ' of K_m^c/K_m by defining $\sigma'(T_n) = T_n$ for every T_n such that $T_n^n = t_m$; i.e. σ' has the same effect as σ on K_{m-1}^c but acts as identity on all the elements T^n . One easily sees that $\sigma = \sigma''\sigma'$ where σ'' is the identity on $K_{m-1}^c\{t_m\}$ and maps each T_n into $\zeta(\sigma, n)T_n$ where $\zeta(\sigma, n)$ is an n th root of unity. Since all roots of unity are in K_0 , it follows that σ'' and σ' commute. Then automorphisms σ' form a subgroup isomorphic to $\hat{Z}$ and σ'' a subgroup isomorphic to $G(K_{m-1}^c/K_{m-1})$. So our proposition follows.

3. The Brauer group of the Amitsur field K_m . Denote the Brauer group of the field L by $B(L)$. The following theorem of Witt [6] is well known: Let L be a local field (i.e. a field which is complete in the topology induced by a discrete, nonarchimedean valuation) with residue class field $\bar{L}$ which is perfect. Then the following sequence, $\{1\} \rightarrow B(L) \rightarrow B(\bar{L}) \rightarrow \chi(G(\bar{L})) \rightarrow \{1\}$, is exact and split. In what follows we use induction on m and the theorem of Witt to determine the structure of $B(K_m)$.

3.1 THEOREM. $B(K_m) \cong \bigoplus_{i=1}^{(m-1)m/2} Q/Z$ for every nonnegative integer m .

PROOF. Since K_0 is algebraically closed and K_1 is a c_1 -field, hence $B(K_0) = B(K_1) = \{1\}$ (see [5, p. 170]). Suppose that $m > 1$. Since $K_m = K_{m-1}$ with respect to the valuation $|\cdot|_m$, it follows from the theorem on Witt that

$$B(K_m) \cong B(K_{m-1}) \oplus \chi(G_c(K_{m-1})).$$

By Proposition 2.3, $\chi(G_c(K_{m-1})) \cong \bigoplus_{i=1}^{m-1} Q/Z$. By the induction hypothesis, $B(K_{m-1}) \cong \bigoplus_{i=1}^{(m-2)(m-1)/2} Q/Z$. Therefore

$$B(K_m) \cong \bigoplus_{i=1}^{(m-2)(m-1)/2} Q/Z \oplus \bigoplus_{i=1}^{m-1} Q/Z \cong \bigoplus_{i=1}^{(m-1)m/2} Q/Z. \quad \text{Q.E.D.}$$

4. The generators of $B(K_m)$. Let L be a field which contains an n th primitive root of unity ζ_n and $\alpha, \beta \in L$. Let $(\alpha, \beta)_n$ be the cyclic L -algebra of degree n which is generated by χ, y with $\chi^n = \alpha, y^n = \beta$ and $\chi y = \zeta_n y \chi$.

Let u, v be two central simple algebras over L . If u, v represent the same element in $B(L)$, then we write $u \sim v$.

4.1 LEMMA. Let $(\alpha, \beta)_n, (\alpha, \gamma)_n$ be two cyclic L -algebras. Then

$$(\alpha, \beta)_n \times (\alpha, \gamma)_n \sim (\alpha, \beta\gamma)_n.$$

PROOF. See [1, Theorem 11, p. 74].

4.2 PROPOSITION. Let L be a cyclic extension of degree n over K_m . Then there exists $t \in L$ such that $t^n = t_1^{\gamma_1} t_2^{\gamma_2} \cdots t_m^{\gamma_m}$ where $\gamma_i \in \mathbb{Z}$ and $L = K_m(t)$.

PROOF. Since K_m contains n th roots of unity, it follows from Hilbert's Theorem 90 that there exists $u \in L$ such that $u^n \in K_m$ and $L = K_m(u)$. Let $u^n = \sum_{i=\gamma_m}^{\infty} u_i t_m^i$ where $u_i \in K_{m-1}$, $u_{\gamma_m} \neq 0$.

$$u^n = u_{\gamma_m} t_m^{\gamma_m} \left(1 + \sum_{i=1}^{\infty} u_{\gamma_m}^{-1} u_{\gamma_m+1} t_m^i \right).$$

By Lemma 2.1, there exists $q\{t\} \in K_m$ such that $q\{t\}^n = 1 + \sum_{i=1}^{\infty} u_{\gamma_m+i} t_m^i$. Choose $v = q\{t\}^{-1}u$, then $v^n = u_{\gamma_m} t_m^{\gamma_m}$ and $L = K_m(v)$. Now $u_{\gamma} \in K_{m-1}$. By using the same method successively, it follows that there exist $t \in L$ such that $L = K_m(t)$ and $t^n = t_1^{\gamma_1} t_2^{\gamma_2} \cdots t_m^{\gamma_m}$.

4.3 THEOREM. $\{(t_i^{\gamma}, t_j)_n \mid 1 \leq i < j \leq m, 1 \leq \gamma < n, n \in \mathbb{Z}^+\}$ is a set of generators of $B(K_m)$.

PROOF. Use induction on m . If $m=0, 1$, then $B(K_0) = B(K_1) = \{1\}$, the theorem is proved. Suppose $m > 1$ and A is a central simple algebra over K_m . It follows from a theorem of Witt [6] that there exists a central simple algebra B over K_{m-1} and a cyclic algebra C over K_m such that $A \sim B \cdot C$ where C is generated by τ, σ with $\tau^n = t_m$, $K_{m-1}(\sigma)$ is a cyclic extension of degree n over K_{m-1} and $\tau\sigma = \zeta_n \sigma\tau$ where ζ_n is an n th root of unity. By Proposition 4.3, σ can be chosen such that $\sigma^n = t_1^{\gamma_1} t_2^{\gamma_2} \cdots t_{m-1}^{\gamma_{m-1}}$, $\gamma_i \in \mathbb{Z}$. Thus,

$$C = (t_1^{\gamma_1} t_2^{\gamma_2} \cdots t_{m-1}^{\gamma_{m-1}}, t_m)_n.$$

By Lemma 4.1, $C \sim \bigtimes_{i=1}^{m-1} (t_i^{\gamma_i}, t_m)_n$. On the other hand, it follows from the induction hypothesis that B can be written as a product of $(t_i^{\gamma}, t_j)_n$ with $1 \leq i < j \leq m-1$, $1 \leq \gamma < n$ and $n \in \mathbb{Z}^+$. Hence the theorem is proved. Q.E.D.

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