CONDITIONS FOR $\text{Ker}(R[X] \rightarrow R[c/b])$ TO HAVE A LINEAR BASE

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ABSTRACT. A number of necessary and/or sufficient conditions are given for the kernel of the homomorphism in the title to be generated by linear polynomials. Two applications are given.

1. Introduction. Let b and c be regular elements (non-zero-divisors) in a ring R (commutative with identity), let X be an indeterminate, and let $K = \text{Ker}(R[X] \rightarrow R[c/b])$ be the kernel of the natural homomorphism from $R[X]$ onto $R[c/b]$. It is known [2, (11.13)] that K has a *linear base*, if R is an integrally closed integral domain; that is, K is generated by linear polynomials, namely by $B = \{dX - e; d, e \in R \text{ and } 0 \neq be = cd\}$. The purpose of this article is to give a number of necessary and/or sufficient conditions for this to happen in the general situation. Since extension rings of the form $R[c/b]$ are frequently encountered in commutative algebra (for example, in the chain problem for prime ideals [2, Example 2, pp. 203–205], in characterizing Noetherian H -rings [5, Proposition 4.7], and in generalizations of Zariski's main theorem [3]), knowledge of such conditions is both desirable and useful, since it can often be used to simplify and/or shorten certain proofs (see Application 2 below).

In the recent commutative algebra conference at Lawrence, Kansas, I. Kaplansky mentioned that [2, (11.13)] was one of the ten “non-Noetherian” theorems which came readily to mind. The conditions given in this paper readily show this non-Noetherian character; and, perhaps not surprisingly, almost all computations in this article are of a very simple character.

2. The following notation is fixed throughout this article. Let R, b, c, X, K , and B be as in the introduction, let R' be the integral closure of R , and, for each $n > 0$, let $T_n = (b, c)^n \cap (b^{n+1}R : cR)$. Also, for a principal ideal dR , $(dR)_a$ will denote its *integral closure* in R (see [2, Exercises p. 34]).

Throughout this paper, implicit use is often made of the fact that b and c are regular elements in R .

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THEOREM 1. *The following statements are equivalent:*

- (1.1) $K = BR[X]$; that is, $\text{Ker}(R[X] \rightarrow R[c/b])$ has a linear base.
- (1.2) $(K, X)R[X] \cap R = cR : bR$.
- (1.3) $T_n \subseteq b^n R$, for all $n > 0$.

PROOF. (1.1) implies $(K, X)R[X] \cap R = (B, X)R[X] \cap R =$ (by definition of B) $(cR : bR, X)R[X] \cap R = cR : bR$, so (1.1) $\rightarrow$ (1.2).

(1.2) $\rightarrow$ (1.3). Let $t \in T_n$, so $t = r_{n+1}c^n + \cdots + r_1b^n$ and $tc/b^{n+1} = -r_0 \in R$. Let $f(X) = r_{n+1}X^{n+1} + \cdots + r_1X + r_0$, so $b^{n+1}f(c/b) = ct + r_0b^{n+1} = 0$. Therefore $f(X) \in K$, so $r_0 \in (K, X)R[X] \cap R = cR : bR$, and so $(tc/b^{n+1})b \in cR$. Thus $t \in b^n R$, hence $T_n \subseteq b^n R$.

(1.3) $\rightarrow$ (1.1). Let $f(X) = r_nX^n + \cdots + r_0 \in K$. That $f(X) \in BR[X]$ will be proved by induction on n , and it may be assumed that $n > 1$. Let $t = r_nc^{n-1} + \cdots + r_1b^{n-1}$. Then $0 = b^n f(c/b) = r_nc^n + \cdots + r_1cb^{n-1} + r_0b^n = ct + r_0b^n$, so $ct = -r_0b^n$, hence $t \in b^n R : cR$. Therefore $t \in T_{n-1} \subseteq b^{n-1} R$ (by (1.3)), so $-r_0b/c = t/b^{n-1} \in R$. Hence $g(X) = r_nX^{n-1} + \cdots + r_2X + (r_1 + r_0(b/c)) \in K$, and so $f(X) = Xg(X) - ((r_0b/c)X - r_0) \in BR[X]$, by induction. Therefore $K = BR[X]$. Q.E.D.

The following two corollaries give some further necessary and sufficient conditions for K to have a linear base.

COROLLARY 2. *The following statements are equivalent:*

- (2.1) $\text{Ker}(R[X] \rightarrow R[c/b])$ has a linear base.
- (2.2) $T_n = b^n(bR : cR)$, for all $n > 0$.
- (2.3) $\bigcup (T_n : b^{n-1}R) = b(bR : cR)$.
- (2.4) $((b, c)^n : b^{n-1}R) \cap (b^2R : cR) \subseteq bR$, for all $n > 0$.

PROOF. (2.1) $\rightarrow$ (1.3) (Theorem 1), and (1.3) implies $T_n = T_n \cap b^n R = b^n(T_n : b^n R) = b^n(bR : cR)$, so (2.1) $\rightarrow$ (2.2).

Clearly (2.2) $\rightarrow$ (2.3) $\rightarrow$ (2.4). Clearly, (2.4) implies $T_1 \subseteq bR$, so let $n > 1$ and assume $T_{n-1} \subseteq b^{n-1}R$. Then, since $T_n \subseteq T_{n-1}$, $T_n = b^{n-1}(T_n : b^{n-1}R) \subseteq$ (by (2.4)) $b^n R$. Therefore, by induction, (2.4) $\rightarrow$ (1.3); and (1.3) $\rightarrow$ (2.1), by Theorem 1. Q.E.D.

COROLLARY 3. *The following statements are equivalent:*

- (3.1) $\text{Ker}(R[X] \rightarrow R[c/b])$ has a linear base.
- (3.2) $(c/b)R[c/b] \cap R = cR : bR$.
- (3.3) $R[c/b]/(c/b)R[c/b] = R/(cR : bR)$.
- (3.4) $\bigcup (c(b, c)^n : b^{n+1}R) = cR : bR$.

PROOF. (3.1) $\Leftrightarrow$ (1.2), by Theorem 1; (1.2) $\Leftrightarrow$ (3.2), by the definition of K ; and, clearly (3.2) $\Leftrightarrow$ (3.3). Also (3.2) $\Leftrightarrow$ (3.4), since a straightforward computation shows that $(c/b)R[c/b] \cap R = \bigcup (c(b, c)^n : b^{n+1}R)$. Q.E.D.

REMARKS. (4.1) All the statements in Theorem 1 and Corollaries 2 and 3 hold when $R=R'$, since then (1.1) holds [2, (11.13)]. (A proof of [2, (11.13)] which holds even when R is not an integral domain can be given using the fact that if $f(X) \in R[X]$ and a is in an extension ring of R such that $f(a)=0$; then ra is integral over R , for some $r \in R$. However, this will not be done in this paper.)

(4.2) If $T_n \subseteq b^n R$, then $T_i \subseteq b^i R$, for $i=1, \dots, n$, since $T_i \subseteq T_n: b^{n-i} R \subseteq b^n R: b^{n-i} R = b^i R$. Therefore, $K=BR[X]$ if and only if $T_n \subseteq b^n R$, for all large n ; or, even, for infinitely many n .

(4.3) Note that $\{c(b, c)^n: b^{n+1} R\}_{n \geq 0}$ and $\{T_n: b^{n-1} R\}_{n \geq 1}$ are ascending sequences of ideals. Therefore, $K=BR[X]$ if and only if $c(b, c)^n: b^{n+1} R = cR: bR$, for all large n , by (3.4); and if and only if $T_n: b^{n-1} R \subseteq bR$, for all large n , by (2.3) and (2.4).

(4.4) $T_n \subseteq (b^n R)_a$ (=the integral closure of $b^n R$ in R). For, if $t \in T_n$, then $t \in (b, c)^n R' \cap (b^{n+1} R': cR') \subseteq b^n R'$ (by (4.1)), hence $t \in b^n R' \cap R = (b^n R)_a$ [4, Lemma 1].

(4.5) If $K=BR[X]$ and A is a flat R -algebra, then, by (1.3),

$$\text{Ker}(A[X] \rightarrow A[c/b])$$

has a linear base.

To obtain some further equivalences of $K=BR[X]$, let $K^* = \text{Ker}(R[X] \rightarrow R[b/c])$, let $B^* = \{eX - d; dX - e \in B\}$, and let $T_n^* = (b, c)^n \cap c^{n+1} R: bR$, for each $n > 0$. Also, if $f(X) = r_n X^n + \dots + r_0 \in R[X]$ ($r_i \in R$ and $r_n \neq 0$), then let $f^*(X)$ be the polynomial $r_0 X^n + \dots + r_n = X^n f(1/X)$, and let $d(f)$ and $o(f)$ denote the degree and order of $f(X)$, respectively. (Therefore, $f^*(X) = X^{d(f)} f(1/X)$ and $f(X) = X^{o(f)} f^{**}(X)$.)

The next result shows that K has a linear base if and only if K^* has.

LEMMA 5. $K=BR[X]$ if and only if $K^*=B^*R[X]$.

PROOF. Let $K=BR[X]$ and $f(X) \in K^*$. Then $f^*(X) \in K$ (since $f^*(c/b) = (c/b)^{d(f)} f(b/c) = 0$), so $f^*(X) = \sum b_i(X) p_i(X)$, where $p_i(X) \in R[X]$ and $b_i(X) \in B$ or $b_i(X) = 0$. Hence $f(X) = X^{o(f)} f^{**}(X) = X^{o(f)+d(f^*)} f^*(1/X) =$ (since $g(X) \rightarrow g(1/X)$ is a degree preserving isomorphism of $R[X]$ onto $R[1/X]$) $X^{o(f)+d(f^*)} \cdot \sum b_i(1/X) p_i(1/X) = X^{o(f)+d(f^*)-d(p_i)-1} \cdot \sum b_i^*(X) p_i^*(X) \in B^*R[X]$. By symmetry, the converse also holds. Q.E.D.

REMARK 6. By Theorem 1 and Corollaries 2 and 3, a number of necessary and sufficient conditions for $K^*=B^*R[X]$ can be written down, and then, by Lemma 5, each of these is equivalent to $K=BR[X]$. Thus, the following conditions are equivalent: (1.1) $\text{Ker}(R[X] \rightarrow R[c/b])$ has a linear base; (6.2) $(K^*, X)R[X] \cap R = bR: cR$; (6.3) $T_n^* \subseteq c^n R$, for all $n > 0$; (6.4) $T_n^* = c^n(cR: bR)$, for all $n > 0$; (6.5) $\bigcup (T_n^*: c^{n-1} R) = c(cR: bR)$;

(6.6) $((b, c)^n : c^{n-1}R) \cap (c^2R : bR) \subseteq cR$, for all $n > 0$; (6.7) $(b/c)R[b/c] \cap R = bR : cR$; (6.8) $R[b/c]/(b/c)R[b/c] = R/(bR : cR)$; (6.9) $\bigcup (b(b, c)^n : c^{n+1}R) = bR : cR$.

The following already known sufficient conditions for $K = BR[X]$ follow from (1.3) or (6.3): $bR = R$; $cR = R$; $bR \subseteq cR$; $cR \subseteq bR$; $bR : cR = bR$ (equivalently, $cR : bR = cR$). Also, if $K = BR[X]$ and d is a regular element in R such that $bR : dR = bR$ (resp., $cR : dR = cR$), then it follows using (1.3) (resp., (6.3)) that $\text{Ker}(R[X] \rightarrow R[cd/b])$ (resp., $\text{Ker}(R[X] \rightarrow R[cd/d])$) has a linear base.

Some less obvious sufficient conditions will now be given.

COROLLARY 7. *Each of the following conditions implies*

$$\text{Ker}(R[X] \rightarrow R[c/b])$$

has a linear base.

$$(7.1) \quad cR : bR = cR : b^2R.$$

$$(7.2) \quad (cR)_a : bR = cR : bR.$$

$$(7.3) \quad b^{k+1}R : cR \subseteq b^kR, \text{ for some } k > 0.$$

$$(7.4) \quad \text{There exists } k > 0 \text{ such that } T_k \subseteq b^kR \text{ and, for all } n > k, (b, c)^n : bR \subseteq (b, c)^{n-1}.$$

PROOF. (7.1) implies $cR : bR = cR : b^nR$, for all $n > 1$, and so $K = BR[X]$ follows from (3.4).

(7.2) $\rightarrow$ (3.4) ($\rightarrow K = BR[X]$), since $U = \bigcup (c(b, c)^n : b^{n+1}R) \subseteq (cR)_a : bR$ (since $U = (c/b)R[c/b] \cap R$, as is easily shown, so $U \subseteq (c/b)R'[c/b] \cap R' \cap R =$ (4.1) $(cR' : bR') \cap R = (cR' \cap R) : bR = [4, \text{Lemma 1}] (cR)_a : bR$).

If (7.3) holds, then it can be shown that $b^{n+1}R : cR \subseteq b^nR$, for all $n > 0$, so $K = BR[X]$, by (1.3).

If (7.4) holds, then, for all $n > k$, $T_k = T_n : b^{n-k}R$ and $T_n = T_n \cap bR = b(T_n : bR) = bT_{n-1}$, so, by induction on $n > k$, $T_n \subseteq b^nR$. Hence, $K = BR[X]$, by (4.2). Q.E.D.

REMARKS. (8.1) (7.1) shows that if R is Noetherian, then $K_n = \text{Ker}(R[X] \rightarrow R[c/b^n])$ has a linear base, for all large n . Also, (7.1) implies that K_n has a linear base, for all $n > 0$ (by the same proof).

(8.2) (7.2) implies that $\text{Ker}(R[X] \rightarrow R[cd/b])$ has a linear base, for all regular elements d in R (by the same proof). It should also be noted that (7.2) includes the special cases: $(cR)_a = cR$; and, $R = R'$ (for then $(cR)_a = cR$). Further, if $(cR)_a = cR$, then $\text{Ker}(R[X] \rightarrow R[c/b])$ has a linear base, for all regular elements $b \in R$.

(8.3) (7.3) implies that $\text{Ker}(R[X] \rightarrow R[c/b^n])$ has a linear base, for all $n > 0$ (by the same proof). Also, (7.3) includes as special cases: $bR : cR = bR$; and, there exists $k > 0$ such that $T_k \subseteq b^kR$ and $b^{k+1}R : cR \subseteq (b, c)^k$ (since then $b^{k+1}R : cR = T_k \subseteq b^kR$).

(8.4) Four further sufficient conditions for $K = BR[X]$ are obtained by interchanging the roles of b and c in Corollary 7 (by Lemma 5). Specifically: $bR : cR = bR : c^2R$; $(bR)_a : cR = bR : cR$; $c^{k+1}R : bR \subseteq c^kR$, for some $k > 0$; there exists $k > 0$ such that $T_k^* \subseteq c^kR$ and, for all $n > k$, $(b, c)^n : cR \subseteq (b, c)^{n-1}$.

COROLLARY 9. *The following statements are equivalent:*

(9.1) $\text{Ker}(R[X] \rightarrow R[c/b])$ has a linear base.

(9.2) $(cS : bS) \cap R = cR : bR$, where $S = R[c/b] \cap R'$.

(9.3) $(bT : cT) \cap R; bR : cR$, where $T = R[b/c] \cap R'$.

PROOF. By Lemma 5, it suffices to prove $(9.1) \Leftrightarrow (9.2)$. For this, $(9.1) \rightarrow (3.2)$, and $(3.2) \rightarrow (9.2)$, since $R[c/b] = S[c/b]$ implies $cR : bR \subseteq (cS : bS) \cap R \subseteq (c/b)S[c/b] \cap S \cap R = (\text{by (3.2)}) cR : bR$. Conversely,

$$(bS)_a : cS = bS : cS,$$

by the definition of S and [6, Remark (ii) following Lemma 2.9]. (Since [6, Lemma 2.4] holds for all rings, [6, Remark (ii)] is applicable.) Therefore, applying (8.4) and (3.2) to S , $(c/b)S[c/b] \cap S = cS : bS$. Hence since $S[c/b] = R[c/b]$, (9.2) implies (3.2); and $(3.2) \rightarrow (9.1)$. Q.E.D.

COROLLARY 10. *Each of the following conditions implies*

$$\text{Ker}(R[X] \rightarrow R[c/b])$$

does not have a linear base:

(10.1) *There exists a m.c. set S in R such that $x \in R'_S, \notin R_S$, where $x \in \{c/b, b/c\}$.*

(10.2) $c^nR : b^nR \not\subseteq cR : bR$, for some $n > 1$.

(10.3) $bR : cR + cR : bR \neq R$ and there is a polynomial in K with a unit coefficient.

(10.4) $bR : cR + cR : bR$ is contained in a height one prime ideal p in R such that $\text{height } pR[X] = 1$.

PROOF. (10.1) implies $(bR : cR)R_S \neq R_S$ and there exists $f(X)$ in $K_S = \text{Ker}(R_S[X] \rightarrow R_S[c/b])$ such that $f(X)$ is monic and $\deg(f) > 1$, so $K_S \neq BR_S[X]$. Therefore $K \neq BR[X]$, by (4.5).

Since $c^nR : b^nR \subseteq c(b, c)^{n-1} : b^nR$, if (10.2) holds, then $K \neq BR[X]$, by (3.4).

Since $BR[X] \subseteq (bR : cR + cR : bR)R[X]$, if (10.3) holds, then $K \neq BR[X]$.

If (10.4) holds, then $BR[X] \subseteq pR[X] \cap K \subset K$. Q.E.D.

APPLICATION 1. Let I be an ideal in a ring R such that I contains a regular element, let t be an indeterminate, and let $A = R[\{ti\}_{i \in I}]$. Then $K = \text{Ker}(A[X] \rightarrow A[1/t])$ has a linear base. This easily follows from (1.3), since A is a graded ring and $T_n = (tb, b)^n \cap (tb)^{n+1}A : bA$ is a homogeneous

ideal, where b is a regular element in I . (It also immediately follows from (3.2) and Lemma 5, since $A[t] = R[t]$ and $tR[t] \cap A = tI = \{ti; i \in I\} = tbA : bA$ with b as above.) Therefore, $(1/t)A[1/t] \cap A = (3.2) bA : tbA = IA$. (The rings A and $A[1/t]$ are called the *restricted Rees ring* and the *Rees ring of R with respect to I* .) This example shows that, for a given $n > 0$, there exists a K with a minimal base of n linear polynomials: namely, choose I with a minimal base of n elements.

APPLICATION 2 (CF. [3, LEMMA 2] AND [1, LEMMA 2(2)]). Let R' be the integral closure of a ring R in its total quotient ring T , let $A = R[c/b] \subseteq T$, and assume $R' \cap A = R$. If there exists a prime ideal P in A such that P is isolated over $p = P \cap R$ (that is, P is maximal and minimal in the set of prime ideals in A lying over p), then $A_P = R_p$ and there exists a regular element $d \in bR : cR$, $\notin p$ such that $R_d = A_d$.

PROOF. By [6, Remark (ii) following Lemma 2.9], $(bR)_d : cR = bR : cR$. Therefore $\text{Ker}(R[X] \rightarrow R[c/b])$ has a linear base, by (8.4). Now P isolated over p implies $C = bR : cR + cR : bR \not\subseteq p$ (since otherwise $K = BR[X] \subseteq CR[X] \subseteq pR[X] \subseteq (p, X)R[X]$). Hence c/b or $b/c \in R_p$, and it is easily seen that $c/b \in R_p$, so $R_p = A_P$. Thus $bR : cR \not\subseteq p$, so there exists a regular element $d \in bR : cR$, $\notin p$, and so $R_d = A_d$, since $A = R[e/d]$, where $e \in R$ is such that $eb = dc$. Q.E.D.

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