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PROJECTIVE COMPACT DISTRIBUTIVE TOPOLOGICAL LATTICES[1]

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ABSTRACT. In the category of all compact distributive topological lattices and their continuous lattice-homomorphisms, it is shown that every projective object is either zero-dimensional or not I-compact.

By a topological lattice we mean a lattice together with a Hausdorff topology under which the two lattice operations are continuous. All terminologies and notation of lattices and category theory used in this paper are the same as in [**2**] and in [**6**], respectively.

Let $\mathscr{L}$ be a category of topological lattices and their continuous lattice-homomorphisms. By a projective object P in $\mathscr{L}$, we mean that, for an onto morphism $f:A\to B$ and a morphism $g:P\to B$ in $\mathscr{L}$, there exists a morphism $h:P\to A$ in $\mathscr{L}$ such that $fh=g$.

Let I be the unit interval $[0, 1]$ of reals with the usual topology and the usual order structure. For a topological lattice L, if L is topologically and algebraically isomorphic with a (closed) sublattice of a product of unit intervals, then we say that L is (I-compact, respectively) I-regular.

LEMMA. *Let $\mathscr{L}$ be a category of topological lattices which is closed hereditary and finitely productive. If P is a connected projective object in $\mathscr{L}$ then, for every prime ideal A of P, either A or $P\backslash A$ is dense in P.*

PROOF. We may assume that $\mathscr{L}$ is nontrivial i.e., $\mathscr{L}$ has at least one nondegenerate object. Then the two element chain lattice $2=\{0, 1\}$ with the discrete topology is always in $\mathscr{L}$. Clearly, the closures A^- and $(P\backslash A)^-$ are both closed sublattices of P. Let $Q=(A^-\times\{0\})\cup((P\backslash A)^-\times\{1\})$. Then Q is a closed sublattice of $P\times 2$.

Now let j be the inclusion of Q into $P\times 2$, and let p be the projection of $P\times 2$ onto P. Then $pj:Q\to P$ is onto. Since P is projective, for pj and the identity i of P, there exists a morphism $f:P\to Q$ in $\mathscr{L}$ such that $pjf=i$. Since P is connected, either $f(P)\subset A^-\times\{0\}$ or $f(P)\subset(P\backslash A)^-\times\{1\}$. If

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$f(P)\subset A^{-}\times\{0\}$, then $P=f^{-1}(A^{-}\times\{0\})$. On the other hand, we can show that $f^{-1}(A^{-}\times\{0\})=A^{-}$. It suffices to show that $f^{-1}(A^{-}\times\{0\})\subset A^{-}$. Let $x\in f^{-1}(A^{-}\times\{0\})$. Suppose that $f(x)=(y, 0)\in A^{-}\times\{0\}$. Since $pjf=i$, we have $x=y$. Thus $x\in A^{-}$. Hence A is dense in P. Similarly, for the case that $f(P)\subset(P\backslash A)^{-}\times\{1\}$, $P\backslash A$ is dense in P.

REMARK. With a few additional conditions to those of the above lemma, it can be generalized to some other Hausdorff topological algebras of finite type as follows:

Let $\mathfrak{A}$ be a category of Hausdorff topological algebras of the same finite type which is closed hereditary and finitely productive, and let P be connected projective in $\mathfrak{A}$. If

(i) the two point algebra 2 with the discrete topology is in $\mathfrak{A}$,

(ii) A and $P\backslash A$ are both subalgebras of P and $Q=(A^{-}\times\{0\})\cup((P\backslash A)^{-}\times\{1\})$ is a closed subalgebra of $P\times 2$ then either A or $P\backslash A$ is dense in P.

For example, in the case of Hausdorff topological spaces (as trivial algebras) (i) and (ii) are always true and, in the case of topological semigroups, if A is a prime ideal of P and the two point meet semilattice with discrete topology is in $\mathfrak{A}$, then (i) and (ii) are always true.

THEOREM. *Let $\mathscr{L}$ be a category of topological distributive lattices which is closed hereditary and finitely productive. Then every projective object in $\mathscr{L}$ is either totally disconnected or not I-regular.*

PROOF. Let P be projective in $\mathscr{L}$. Suppose that P is not totally disconnected. Then we have a connected component C of P with more than two points, and it is a closed convex sublattice of P [**4**]. Let $J=[\alpha, \beta]$ be a nondegenerate closed interval of C. Then J is also a closed interval in P, which is connected since C is. Further, it is easy to see that the map $f:P\to J=[\alpha, \beta]$ defined by $f(x)=\alpha\vee(x\wedge\beta)$ is a retraction. Hence J is also projective in $\mathscr{L}$. Now we show that J does not have a nonconstant continuous lattice-homomorphism from J into I. Indeed, if $g:J\to I$ is a nonconstant continuous lattice-homomorphism, then $g(J)=[r, s]\subset I$ with $r<s$. It is easy to see that $f^{-1}([r, t])$ $(r<t<s)$ is a closed prime ideal of J, and it is neither dense in J nor is its complement dense in J. This is a contradiction of the lemma.

COROLLARY 1. *Let $\mathscr{D}$ be the category of all compact distributive lattices. Then every projective lattice in $\mathscr{D}$ is either zero-dimensional or not I-compact.*

It is known [7] that if L is a compact distributive lattice then L is I-compact iff L is completely distributive. Hence by the theorem every projective lattice in the category of all compact completely distributive lattices and their continuous lattice-homomorphisms is zero-dimensional.

It is shown [3] that, in the category of all zero-dimensional compact distributive lattices, P is projective iff P is a retract of the residually finite completion of a free distributive lattice.

Hence we have the following:

COROLLARY 2. *Let $\mathscr{C}\mathscr{D}$ be the category of all compact completely distributive lattices. Then every projective lattice in $\mathscr{C}\mathscr{D}$ is a retract of the residually finite completion of a free distributive lattice.*

REMARK. It is known [5] that there actually exists a compact distributive lattice which is not I-compact. However, the author does not know whether a projective one which is not I-compact exists in $\mathscr{D}$. If such a projective P exists in $\mathscr{D}$, then P must have the following properties (i)–(iii):

(i) P has a nondegenerate connected retract which has no nonconstant continuous lattice-homomorphism into I.

(ii) P/ρ, where $x\rho y$ iff x and y belong to the same connected component of P, is projective in $\mathscr{C}\mathscr{D}$.

(iii) If P is connected then, for any upper (or lower) bound x of a nonempty open subset of P, $x \vee P$ (or $x \wedge P$ respectively) has void interior.

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