COHOMOLOGY OF FIBRE SPACES WITH GROUP BUNDLE COEFFICIENTS

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ABSTRACT. The purpose of this paper is to construct a resolution of a group bundle γ , a classifying spectrum for a cohomology functor $H^*(\ ; \gamma)$ (coefficients in γ) defined on a category of fibre spaces, and to clarify and note some implications of the close relationship between these two constructions.

1. Introduction. A group bundle, as used in this paper, consists of two Hausdorff k -spaces $E(\gamma)$ (the total space) and B (the base space), a continuous projection $E(\gamma) \rightarrow B$, and a fibre preserving map from the fibre product (in the category of k -spaces) $\gamma \times \gamma$ to γ that induces on each fibre of γ an abelian group structure. Further, the 0-section and the inversion map are required to be continuous (see [2], [3], [4]). This notion of group bundle differs from the usual notion in that forming fibre products in the sense of k -spaces allows for an increase in the number of group bundle structures on γ . If, for example, γ is locally compact the two notions coincide. Define $H^p(B; \gamma) = H^p(B; \tilde{\gamma})$ where $\tilde{\gamma}$ is the sheaf of germs of sections of γ and call γ acyclic if $\tilde{\gamma}$ is acyclic (see [1]).

2. Definitions and remarks. An exact sequence of group bundles $0 \rightarrow \gamma_0 \rightarrow \gamma_1 \xrightarrow{i_1} \gamma_2 \rightarrow \cdots$ (over a fixed base B) is a *resolution* of γ_0 if the associated sequence of sheaves $0 \rightarrow \tilde{\gamma}_0 \rightarrow \tilde{\gamma}_1 \rightarrow \cdots$ is exact. The resolution is *acyclic* if each γ_n is acyclic. By [1, p. 34], $H^n(B; \gamma_0) \simeq H^n(C(\tilde{\gamma}_0))$ where $C(\tilde{\gamma}_0)$ is the cochain complex of sheaf sections: $0 \rightarrow \Gamma(\tilde{\gamma}_0) \rightarrow \Gamma(\tilde{\gamma}_1) \rightarrow \cdots$ associated to an acyclic resolution of γ_0 . Define γ_n to be the image of i_n . Since $C(\tilde{\gamma}_0)$ is isomorphic to the complex of bundle sections

$$C(\gamma_0): 0 \rightarrow \Gamma(\gamma_0) \xrightarrow{\Gamma(i_1)} \Gamma(\gamma_1) \rightarrow \cdots$$

and

$$\text{Ker}(\Gamma(i_{n+1})) = \Gamma(\gamma_n),$$

there is a natural isomorphism $H^n(B; \gamma_0) \simeq \Gamma(\gamma_n) / \text{Im } \Gamma(i_n)$. This result can be interpreted as

2.1. $H^n(B; \gamma_0)$ is naturally isomorphic to the group of equivalence

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classes of sections of γ_n , where two sections are identified if and only if their difference lifts (by i_n) to a section of γ_{n-1} ($i_0=0$).

Further, the sequence $0 \rightarrow \gamma_n \rightarrow \nu_n \rightarrow \nu_{n+1} \rightarrow \dots$ is clearly an acyclic resolution, hence

$$2.2. \quad H^p(B; \gamma_n) \simeq H^p(C(\gamma_n)) \simeq H^{p+n}(B; \gamma_0) \text{ to } (n \geq 0, p > 0).$$

3. An acyclic resolution. A group bundle is said to be an $(L)NDR$ group bundle if (locally) the 0-section is a vertical deformation retract of an open neighborhood; see [3], [4, 6.2]. In [3] the following is proved.

LEMMA. *If γ_0 is an $(L)NDR$ group bundle, then there is an exact sequence of group bundles (depending functorially on γ_0).*

$$3.1. \quad 0 \rightarrow \gamma_0 \xrightarrow{i} \nu_0 \xrightarrow{j} \gamma_1 \rightarrow 0 \text{ for which}$$

$$3.2. \quad \gamma_1 \text{ is an } (L)NDR \text{ group bundle.}$$

3.3. $j: E(\nu_0) \rightarrow E(\gamma_1)$ has local sections, i.e. there is an open cover $\{U_\alpha\}$ of $E(\gamma_1)$ and maps s_α such that $js_\alpha = \text{id}$ on U_α .

$$3.4. \quad \nu_0 \text{ is shrinkable to the 0-section.}$$

3.5. j has the covering homotopy property for vertical homotopies $H_i: X \rightarrow E(\gamma_1)$ if γ_0 is NDR ($LNDR$ if X is paracompact).

Property 3.2 implies a long exact sequence $0 \rightarrow \gamma_0 \rightarrow \nu_0 \xrightarrow{i_1} \nu_1 \rightarrow \dots$ can be constructed inductively by applying the lemma to γ_1 and splicing the resulting sequence 3.1 to the sequence 3.1 associated to γ_0 , etc. Since “ $\sim$ ” is left exact (by construction [3] γ_0 has topology induced by i) and j is onto by 3.3 (lift germs of sections of γ_1 by an appropriate s_α) the sequence so obtained is a resolution, hereafter called the *canonical resolution*.

If B is paracompact 3.4 implies $\tilde{\nu}_0$ is soft (thus acyclic, [1, p. 49], i.e., every section s over a closed set K extends to B . Indeed, there is an open set $U \supset K$ and $\sigma' \in \Gamma(\nu_0|U)$ such that $\tilde{\sigma}'|K = s$. Define $\sigma \in \Gamma(\nu_0)$ by

$$\begin{aligned} \sigma(b) &= 0 & \text{if } \tau(b) \leq \tfrac{1}{2}, \\ &= H_{2\tau(b)-1}(\sigma'(b)) & \text{if } \tau(b) \geq \tfrac{1}{2}, \end{aligned}$$

where H_t is a vertical homotopy shrinking ν_0 (3.4) ($H_0=0$, $H_1=\text{id}$), $\tau: B \rightarrow I$ is such that $\tau^{-1}(1) \supset \bar{U}_1$, $\tau|(B-U)=0$ where U_1 is open and $U \supset \bar{U}_1 \supset U_1 \supset K$. Clearly $\sigma|U_1 = \sigma'|U_1$, thus $\tilde{\sigma}$ extends s . This proves

3.6. THEOREM. *Any $LNDR$ group bundle over a (paracompact) k -space has a (acyclic) resolution.*

Assume B is paracompact for the rest of this section. The construction of this resolution implies that

$$0 \rightarrow \gamma_{n-1} \rightarrow \nu_{n-1} \xrightarrow{i_n} \gamma_n \rightarrow 0$$

satisfies 3.4, and 3.5 if γ_0 is $LNDR$. This gives a “homotopic” description

of the "algebraic" equivalence relation on sections of γ_n (2.1); namely, for $s_0, s_1 \in \Gamma(\gamma_n)$ there is $s \in \Gamma(\gamma_{n-1})$ such that $i_n s = s_1 - s_0$ if and only if s_0 and s_1 are vertically homotopic. Indeed, if $i_n s = s_1 - s_0$ define the homotopy by $H'_t(b) = i_n(H_t(s(b))) + s_0(b)$ where H_t is a shrinking of γ_{n-1} ($H_0 = 0$, $H_1 = \text{id}$). Conversely, given an H'_t ($H'_0 = s_0$, $H'_1 = s_1$) then $H''_t = H'_t - s_0$ is a vertical homotopy between 0 and $s_1 - s_0$ such that H''_0 is covered by the 0-section of γ_{n-1} . By 3.5, H''_1 is covered by a section s and $i_n s = s_1 - s_0$, thus

3.7. If γ_0 is *LNDR* there is a natural isomorphism between $H^n(B; \gamma_0)$ and the group of equivalence classes of vertically homotopic sections of γ_n .

4. **A classifying spectrum.** For B a k -space let P_B be the category of paracompact k -fibre spaces ([2], [3]) over B ($\xi \in P_B$ means $E(\xi)$ is a paracompact k -space). For γ_0 a group bundle on B define $H^p(\xi; \gamma_0)$ (p th cohomology of ξ with coefficients in γ_0) as $H^p(E(\xi); \xi(\gamma_0))$ where the group bundle $\xi(\gamma_0)$ is the pullback (in category of k -spaces) of γ_0 via the projection $P(\xi)$ of ξ . It is not hard to see that conditions 3.1–3.5 are invariant under pullback (the pullback by $P(\xi)$ of 3.1 is an exact sequence on $E(\xi)$ satisfying 3.2–3.5). Consequently the pullback of the canonical resolution of γ_0 is an acyclic resolution of $\xi(\gamma_0)$. Since pullback induces an isomorphism between $\Gamma(\xi(\gamma_n))$ and $\text{Hom}(\xi, \gamma_n)$ (the set of fibre preserving maps $\xi \rightarrow \gamma_n$), 2.1 implies

4.1. If γ_0 is an *LNDR* then $H^*(\xi; \gamma_0)$ is the cohomology of the complex $\{\text{Hom}(\xi, \gamma_n), \text{Hom}(i_{n+1})\}$.

Result 3.7 (with $(E(\xi); \xi(\gamma_n))$ replacing $(B; \gamma_0)$) plus the fact that pull back induces an isomorphism between the group of equivalence classes of vertically homotopic sections of $\xi(\gamma_n)$ and the group of equivalence classes of vertically homotopic fibre maps $\xi \rightarrow \gamma_n$, denoted by $[\xi, \gamma_n]$, gives

4.2. **THEOREM.** For γ_0 an *LNDR* the image sequence $\{\gamma_1, \gamma_2, \dots\}$ of the canonical resolution of γ_0 is a classifying spectrum for the functor $H^*(\ ; \gamma_0)$ defined on P_B , i.e., $H^n(\xi; \gamma_0) \simeq [\xi, \gamma_n]$, $n > 0$.

The following is proved in [3].

4.3. If γ_0 is *NDR* then $[\xi, \gamma_1]$ and the group of isomorphism classes of numerable principal γ_0 bundles on ξ are naturally isomorphic. Normality of $E(\xi)$ is not needed, but if $E(\xi)$ is paracompact, numerable can be omitted and *NDR* replaced by *LNDR*. A similar interpretation holds for $[\xi, \gamma_n]$.

Combining 4.2, 2.2 and 4.3 gives

4.4. **THEOREM.** If γ_0 is *LNDR* then $H^p(\xi; \gamma_n)$ is naturally isomorphic to the group of isomorphism classes of principal γ_{p+n-1} bundles on ξ for $\xi \in P_B$ ($n \geq 0$, $p > 0$).

Besides providing a geometric interpretation of $H^*(; \gamma_0)$ this implies ($p=1$) that γ_n is the classifying group bundle (see [3]) of γ_{n-1} (actually

$$E(\gamma_n) \xrightarrow{i_n} E(\gamma_n) \rightarrow B$$

is a universal bundle for γ_{n-1}).

REFERENCES

1. G. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967. MR 36 #4552.
2. A. Grothendieck, *A general theory of fibre spaces with structure sheaf*, NSF Research Project Report No. 4, University of Kansas, Lawrence, Kan., 1958.
3. M. Mielke, *Universal fibre bundles of group bundles* (to appear).
4. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. 14 (1967), 133-152. MR 35 #970.

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