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LOCAL BOUNDEDNESS AND CONTINUITY FOR A FUNCTIONAL EQUATION ON TOPOLOGICAL SPACES[1]

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ABSTRACT. It is known that the locally bounded solutions f of Cauchy's functional equation $f(x)+f(y)=f(x+y)$ on the reals are necessarily continuous. We shall extend this result to the functional equation $f(x)+g(y)=h(T(x, y))$ on topological spaces.

1. **Introduction.** Let X, Y be topological spaces and let $f: X\to R$ (the reals), $g: Y\to R$, $T: X\times Y\to R$ and $h: T(X\times Y)\to R$ be functions satisfying the functional equation

$$f(x)+g(y)=h(T(x,y)) \tag{1}$$

for all $x\in X$, $y\in Y$. We shall give some sufficient topological assumptions on X and T so that the local boundedness and nonconstancy of f insure that g is continuous. The method was suggested by the work of J. Pfanzagl in his paper [**6**] generalizing a result of G. Darboux [**2**].

2. **Main theorems.**

THEOREM 1. *For equation* (1), *if each pair of points of X is contained in the continuous image of some connected and locally connected space* (*for instance, when X is connected and locally connected or when X is pathwise connected*), *T is continuous in each of its two variables and f is nonconstant and locally bounded from above* (*or from below*) *at each point of X, then g is continuous on Y.*

PROOF. Let $a, b\in X$ be such that $f(a)\neq f(b)$. There exist a connected and locally connected space $\tilde{X}$ and a continuous mapping $\gamma:\tilde{X}\to X$ such that $a, b\in\gamma(\tilde{X})$. The functions $\tilde{f}:=f\circ\gamma$ and $\tilde{T}$ with $\tilde{T}(\tilde{x},y):=T(\gamma(\tilde{x}),y)$ for $\tilde{x}\in\tilde{X}$, $y\in Y$, now satisfy the induced functional equation

$$\tilde{f}(\tilde{x})+g(y)=h(\tilde{T}(\tilde{x},y)) \tag{$\tilde{1}$}$$

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for all $\tilde{x} \in \tilde{X}$, $y \in Y$. The local boundedness of f passes to $\tilde{f}$ and the continuity of T in each variable passes to $\tilde{T}$. With this observation there is no loss of generality if we suppose from the very beginning that X is connected and locally connected.

Since X is connected and f is nonconstant on X, f cannot be locally constant on X and there exists a point $e \in X$ such that f is nonconstant on every neighbourhood of e. As X is locally connected and f is locally bounded from above at e there exists an open connected neighbourhood U of e on which f is bounded from above. Thus f is nonconstant and bounded from above on the connected and locally connected set U.

Let x_1, $x_2 \in U$ be such that $f(x_1) \neq f(x_2)$. It follows from equation (1) that $T(x_1, y) \neq T(x_2, y)$ for all $y \in Y$.

Let $y_0 \in Y$ be arbitrarily given and we shall prove the continuity of g at y_0. We may suppose that $t_1 := T(x_1, y_0) < T(x_2, y_0) =: t_2$. By Lemma 1 in Pfanzagl [5] there exists a connected $B \subseteq U$ such that $T(B, y_0) = \,]t_1, t_2[$. Let $\varepsilon > 0$ be arbitrarily given. Since $\sup f(B) < \infty$, there exists $x_0 \in B$ such that $f(x_0) \geqq f(x) - \varepsilon$ for all $x \in B$.

Let $M := \{y \in Y : t_1 < T(x_0, y) < t_2\}$. Then $y_0 \in M$ and, as $T(x_0, \cdot)$ is continuous on Y, M is a neighbourhood of y_0.

For each $y \in M$, $T(x_0, y) \in \,]t_1, t_2[\, = T(B, y_0)$ and so there exists $x \in B$ such that $T(x_0, y) = T(x, y_0)$. Thus $f(x_0) + g(y) = h(T(x_0, y)) = h(T(x, y_0)) = f(x) + g(y_0)$. As $f(x_0) \geqq f(x) - \varepsilon$ we have $g(y) \leqq g(y_0) + \varepsilon$.

Let x_3, $x_4 \in B$ be arbitrarily chosen such that $t_3 := T(x_3, y_0) < T(x_0, y_0) < T(x_4, y_0) =: t_4$.

Let $N := \{y \in Y : T(x_3, y) < T(x_0, y_0) < T(x_4, y)\}$. Then $y_0 \in N$ and, as $T(x_3, \cdot)$ and $T(x_4, \cdot)$ are continuous on Y, N is a neighbourhood of y_0.

For each $y \in N$, $T(B, y)$ is an interval of R as B is connected and $T(\cdot, y)$ is continuous. Furthermore, $T(x_3, y)$ and $T(x_4, y)$ are points of $T(B, y)$ with $T(x_3, y) < T(x_0, y_0) < T(x_4, y)$ and so $T(x_0, y_0) \in T(B, y)$. Hence there exists $x \in B$ such that $T(x_0, y_0) = T(x, y)$. From this we have $f(x_0) + g(y_0) = f(x) + g(y)$. As $f(x_0) \geqq f(x) - \varepsilon$ we have $g(y_0) - \varepsilon \leqq g(y)$.

$M \cap N$ is then a neighbourhood of y_0 and $g(y_0) - \varepsilon \leqq g(y) \leqq g(y_0) + \varepsilon$ for every $y \in M \cap N$. This proves the continuity of g at y_0.

REMARK 1. Lemma 1 in Pfanzagl [5] is given as: let X be a connected and locally connected Hausdorff space, $\theta : X \to R$ a continuous map, then to any t_1, $t_2 \in \theta(X)$ with $t_1 < t_2$, there exists a connected component B of $\theta^{-1}(]t_1, t_2[)$ such that $\theta(B) = \,]t_1, t_2[$. The proof is based on a theorem of Wilder [7, p. 46, Theorem 3.8]. The assumption that X is Hausdorff is however not used and can be removed.

COROLLARY 1. *If X is locally connected, $T : X \times X \to R$ is continuous in each variable, $f : X \to R$ is locally bounded from above (or from below) at* *each point of X and h is any function on $T(X, X)$ satisfying the functional equation*

$$f(x)+f(y) = h(T(x, y)) \tag{2}$$

for all $x, y \in X$, then f must be continuous on X.

PROOF. For a point $a \in X$, if f is locally constant at a then f is continuous at a. We may suppose now f is not locally constant at a and hence there exists an open connected neighbourhood U of a such that f is bounded and nonconstant on U. We can apply Theorem 1 to the equation

$$f(x)+f(y) = h(T(x, y))$$

for all $x \in U$, $y \in X$ yielding the continuity of f on X.

REMARK 2. Corollary 1 is proved by Pfanzagl [6] under stronger assumptions on X—that X is locally compact and locally connected Hausdorff.

THEOREM 2. *For equation* (1), *if each pair of points of X is contained in some compact connected subset of X, T is jointly continuous on the product space $X\times Y$ and f is nonconstant and locally bounded from above on X* (*or locally bounded from below on X*), *then g is continuous on Y.*

PROOF. Similar to the argument given in the first paragraph in the proof of Theorem 1 we may suppose that X is compact and connected. We note that f is then bounded from above on every subset of X.

Let $y_0 \in Y$ and $\varepsilon>0$ be arbitrarily given.

Since f is nonconstant on X, for each $y \in Y$ the function $T(\cdot, y)$ is nonconstant on X and $T(X, y)$ is a proper closed interval of R. Write $T(X, y_0)=[a, b]$ with $a<b$. Let $A=\{x \in X: T(x, y_0)=a\}$, $B=\{x \in X: T(x, y_0)=b\}$ and $C=\{x \in X: a<T(x, y_0)<b\}$. The sets A, B and C partitioned X with A and B being closed in X and therefore compact. Since $\sup f(C)<\infty$ there exists $x_0 \in C$ such that $f(x_0)\geqq f(x)-\varepsilon$ for all $x \in C$.

We first let $M:=\{y \in Y: a<T(x_0, y)<b\}$.

Similar to the proof lines in Theorem 1, M is seen to be a neighbourhood of y_0 and $g(y)\leqq g(y_0)+\varepsilon$ for all $y \in M$.

Secondly, we let $N:=\{y \in Y: T(x, y)<T(x_0, y_0)<T(x', y)$ for all $x \in A$, $x' \in B\}$. We proceed to show that N is a neighbourhood of y_0.

For each $x \in A$ we have $T(x, y_0)=a \in\ ]-\infty, T(x_0, y_0)[$. T is jointly continuous and so there exist neighbourhoods $U(x)$, $V_x(y_0)$ of x and y_0 respectively such that $T(U(x), V_x(y_0))\subseteq\ ]-\infty, T(x_0, y_0)[$. Now, because A is compact, there exists a finite subset $A' \subseteq A$ such that $\bigcup \{U(x): x \in A'\}\supseteq A$. The finite intersection $V:=\bigcap \{V_x(y_0): x \in A'\}$ is then a neighbourhood of y_0 and $T(A, V)\subseteq\ ]-\infty, T(x_0, y_0)[$. Similarly, there exists a neighbourhood W of y_0 such that $T(B, W) \subseteq \,]T(x_0, y_0), \infty[$. Now $N \supseteq V \cap W$ and is a neighbourhood of y_0.

For each $y \in N$, $T(X, y)$ is an interval of R. The fact that $T(A, y) \subseteq \,]-\infty, T(x_0, y_0)[$ and $T(B, y) \subseteq \,]T(x_0, y_0), \infty[$ implies $T(x_0, y_0) \in T(C, y)$. Thus there exists $x \in C$ such that $T(x_0, y_0) = T(x, y)$. It follows that $f(x_0) + g(y_0) = f(x) + g(y)$. Since $f(x_0) \geqq f(x) - \varepsilon$ we have $g(y_0) - \varepsilon \leqq g(y)$.

$M \cap N$ is then a neighbourhood of y_0 and $g(y_0) - \varepsilon \leqq g(y) \leqq g(y_0) + \varepsilon$ for all $y \in M \cap N$. This proves the continuity of g at y_0.

THEOREM 3. *For equation* (1), *if X is connected, T is continuous in each variable and f is nonconstant and bounded on X (from both sides), then g is continuous on Y.*

PROOF. Let $y_0 \in Y$ and $\varepsilon > 0$ be arbitrarily given.

The nonconstancy of f in equation (1) implies that $T(\cdot, y_0)$ is nonconstant. $T(X, y_0)$ is then a nondegenerated interval of R and there exist $t_1, t_2 \in T(X, y_0)$ with $t_1 < t_2$. The set $B = \{x \in X : T(x, y_0) \in \,]t_1, t_2[\,\}$ is mapped by $T(\cdot, y_0)$ onto $]t_1, t_2[$. Since f is bounded from above on B there exists $x_0 \in B$ such that $f(x_0) \geqq f(x) - \varepsilon$ for all $x \in B$. If we set

$$M := \{y \in Y : T(x_0, y) \in \,]t_1, t_2[\,\}$$

we see that M is a neighbourhood of y_0. Furthermore for each $y \in M$, $T(x_0, y) \in \,]t_1, t_2[\, = T(B, y_0)$ and so there exists $x \in B$ such that $T(x_0, y) = T(x, y_0)$. It follows that $f(x_0) + g(y) = f(x) + g(y_0)$. As $f(x_0) \geqq f(x) - \varepsilon$ we have $g(y) \leqq g(y_0) + \varepsilon$.

The above argument applies to the functions $\tilde{f} = -f$, $\tilde{g} = -g$, and $\tilde{h} = -h$ satisfying again equation (1). Hence there exists a neighbourhood $\tilde{M}$ of y_0 such that $\tilde{g}(y) \leqq \tilde{g}(y_0) + \varepsilon$ for all $y \in \tilde{M}$, i.e. $g(y_0) - \varepsilon \leqq g(y)$.

On the neighbourhood $M \cap \tilde{M}$ we have $g(y_0) - \varepsilon \leqq g(y) \leqq g(y_0) + \varepsilon$ for every $y \in M \cap \tilde{M}$. This proves the continuity of g.

3. **Some examples.** The connectedness of X is a common assumption in Theorems 1, 2 and 3. Its essentiality can be seen from the following example.

EXAMPLE 1. We take $X = \{0, 1\}$ the discrete space $\subseteq R$, $Y = R$ the reals with the usual topology, $T(x+y) = x + y$, $f: X \to Y$ the natural inclusion map, $g = h: R \to R$ an additive function of the reals which is continuous at no place and leaving the rationals fixed. Obviously equation (1) is satisfied, X is locally connected and compact, T is jointly continuous and f is bounded, nonconstant on X, while g is continuous at no place.

However, connectedness of X alone is not sufficient to give Theorems 1 and 2. This has been shown by C. Hipp who gave the following example.

EXAMPLE 2 (BY C. HIPP). Let $X=Y=R$, Y endowed with the cannonical topology τ on R and X endowed with the topology τ_1 generated by τ and all subsets of R containing the rational numbers Q. Then (X, τ_1) is connected. Let ϕ be a discontinuous (with respect to τ) solution of the Cauchy equation

$$\phi(x) + \phi(y) = \phi(x + y) \quad \text{for all } x, y \in R.$$

As for each $x \in X$, ϕ is bounded on $(\{x\} \cup Q) \cap (x-1, x+1)$ which is a τ_1 neighbourhood of x, we have the local boundedness of ϕ on (X, τ_1). The map T with $T(x, y)=x+y$ is jointly continuous on $(X, \tau) \times (Y, \tau)$ and hence continuous on $(X, \tau_1) \times (Y, \tau)$. However ϕ is continuous on (Y, τ) at no place.

The local connectedness of X for Corollary 1 is by no means redundant. We illustrate this by the following example.

EXAMPLE 3. We take $X=\{n^{-1}: n=1, 2, \cdots\} \cup \{0\}$ as a subspace of R, $T(x, y)=x+y\sqrt{2}$ on $X \times X$, $f(x)=0$ if $x \neq 0$ and $f(0)=1$, $h(n^{-1})= h(n^{-1}\sqrt{2})=1$ for all $n=1, 2, \cdots$ and $h(n^{-1}+m^{-1}\sqrt{2})=0$ for all $n, m= 1, 2, \cdots$ and $h(0)=2$. Obviously, equation (2) is satisfied, X fails to be locally connected at 0, T is jointly continuous, f is bounded on X but fails to be continuous at 0.

Some uniqueness theorems concerning the continuous solutions of equations (1) and (2) are given in Ng **[4]** and Pfanzagl **[5]**.

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