PRIMITIVE IDEALS IN VON NEUMANN ALGEBRAS¹

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ABSTRACT. For a von Neumann algebra it is shown that the set of primitive ideals containing a fixed maximal ideal of the center is sequentially closed in the order topology defined on the set of all ideals containing the maximal ideal. As a corollary, it is shown that every ideal generated by a sequence of elements of a von Neumann algebra and a maximal ideal of the center is either primitive or simple modulo a primitive ideal.

1. Preliminaries. Let A be a von Neumann algebra with center Z . Let ζ be a maximal ideal of Z and let C_ζ be the set of all (closed two-sided) ideals of A containing ζ . The set C_ζ is linearly ordered by inclusion and therefore every ideal I in C_ζ is *prime*, i.e. $I \supset I'I''$ implies $I \supset I'$ or $I \supset I''$. The set of all prime ideals of A is $\bigcup_\zeta C_\zeta$. It is known that every *primitive* ideal, i.e. the kernel of an irreducible representation of A , is prime. Since every prime ideal is primitive for a separable C^* -algebra [1], the question arises as to whether a prime ideal is primitive for other algebras. We consider some aspects of that question in this paper.

Notation. A is a von Neumann algebra with center Z ; (S) is the set of projections in $S \subset A$; a, b, c are elements of A ; e, f, g are elements of (A) ; p, q are elements of (Z) ; ζ is a maximal ideal of Z ; $[\zeta]$ is the ideal in A generated by ζ .

2. Ideals. Given $e \in (A)$, let $S(e)$ denote the set of all finite sums of orthogonal projections e_i ($1 \leq i \leq n$) in A with $e_i < e$.

LEMMA 1. *If e and f are projections in A such that $e \in S(f)$, then $S(e) \subset S(f)$.*

PROOF. It is sufficient to show that $g \in S(f)$ whenever $g \leq e$. Let e_i ($1 \leq i \leq n$) be orthogonal projections of sum e with $e_i < f$. The range projection g'_i of ge_i satisfies the relations $g'_i \leq g$ and $g'_i < e_i$ [3, III, §1, Proposition 2]. Setting $g''_0 = 0$ and $g''_k = \bigvee \{g'_i \mid 1 \leq i \leq k\}$ for $1 \leq k \leq n$, we get that $\{g''_k - g''_{k-1} \mid 1 \leq k \leq n\}$ implements the relation $g \in S(f)$ since $g''_k - g''_{k-1} < g''_k < e_i < f$ [3, III, §1, Corollary 1, Proposition 2]. Q.E.D.

The next definition was considered in a less general case in [5].

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DEFINITION 2. Let $(I(e))$ be the family of all projections f in A such that $pf' \in [\zeta]$ whenever $pe < pf'$ for any $f' \in S(f)$ and $p \in (Z)$. Let $(J(e))$ be the family of all projections f in A such that $pf < pg$ for some $g \in S(e)$ and $p \in (Z) \setminus \zeta$.

We collect some statements we need in the next lemma.

LEMMA 3. *The following statements are true:*

- (i) $(I(e))$ and $(J(e))$ are the sets of all projections of the ideals $I(e)$ and $J(e)$;
- (ii) $I(e) \supset [\zeta]$;
- (iii) $J(e)$ is generated by e and ζ ;
- (iv) if $e \notin [\zeta]$, then $J(e)$ is the smallest ideal of A properly containing $I(e)$;
- (v) the ideal $J(e)$ is of the form $J(e) = I(f)$ if e is a properly infinite projection not in $[\zeta]$.

PROOF. (i) A short calculation using Lemma 1 and [3, III, §1, Corollary 1, Proposition 2] shows that $(I(e))$ and $(J(e))$ are p -ideals in the sense of F. B. Wright [7] and hence the sets of projections of certain ideals in A .

(ii) and (iii) obvious.

(iv) If $f \in (I(e))$ and if $p \in (Z)$ with $pe < pf$ and $(1-p)f < (1-p)e$, then $pf \in [\zeta]$ and so $f \in J(e)$. Thus we get that $I(e) \subset J(e)$. Conversely, let I be an ideal that properly contains $I(e)$. Then there is an $f \in (I)$, an $f' \in S(f)$, and a $p \in (Z)$ with $pe < pf'$ and $pf' \notin [\zeta]$. Because $pf' \in I$ and $[\zeta] \subset I$, we see that $pe \in I$ and so $J(e) \subset I$. By hypothesis, $e \notin [\zeta]$ and so $I(e) \neq J(e)$.

(v) Since the properly infinite projection e can be expressed as a sum of n ($n=1, 2, \dots$) orthogonal projections equivalent to e , we see that $S(e) = \{f \in (A) \mid f < e\}$. Since $S(e)$ is a p -ideal [7], there is an ideal I in A such that $(I) = S(e)$. If $\{a_i\}$ is a bounded subset of I and $\{p_i\}$ a corresponding orthogonal set in (Z) of sum 1, then $\sum p_i a_i$, which converges in the strong operator topology, is in I . Indeed, the obvious case in which each $a_i \in (I)$ immediately implies the most general case. There is a properly infinite projection $f \notin [\zeta]$ and a $p \in (Z)$ such that $S(e) = \{g \in (Ap) \mid qf < qg \text{ for some } q \in (Z) \text{ implies } qf = 0\}$ [5, Theorem 2.4]. We show $J(e) = I(f)$. Notice that $p \notin \zeta$ since $ep = e$. If $g \in S(e)$ and if $q \in (Z)$ with $qf < qg$, then $qf = 0$ and thus $q \in \zeta$. Hence $e \in I(f)$. By (ii) and (iii) we get $J(e) \subset I(f)$. Conversely, if q is the largest element of (Z) such that $qf < qg$ for given $g \in (I(f))$, then $qg \in [\zeta]$. We also have that $(1-q)pg \in S(e)$. Because $1-p \in [\zeta]$, we get that g is in the ideal $J(e)$ by (ii) and (iii). Thus we have that $I(f) \subset J(e)$. Q.E.D.

We now consider the set $C = C_\zeta$ of prime ideals in A that contain ζ introduced in §1. The set C linearly ordered by inclusion. We consider the usual *order topology* on C .

THEOREM 4. *The set P of primitive ideals in A that contain ζ is a sequentially closed subset of C .*

PROOF. If $\{I_n\}$ is a sequence in P with $\lim I_n = I$, then $\bigvee_m \bigwedge \{I_n | n \geq m\}$. So it is sufficient to show that any strictly increasing (resp. decreasing) sequence $\{I_n\}$ in C has least upper (resp. greatest lower) bound in P .

First we consider the least upper bound. We may assume that $I_0 = [\zeta]$. There are e_n with $I_{n-1} \subset I(e_n) \subset J(e_n) \subset I_n$ by Lemma 3(iv). Because $I = \bigvee I_n = \bigvee I(e_n)$, we may assume that $I_n = I(e_n)$ ($n \geq 1$). Furthermore, we easily verify that $I_n = I(\bigvee \{e_m | m \leq n\})$ and so we may assume that $e_1 \leq e_2 \leq \dots$. Since each proper ideal of A is the intersection of primitive ideals [2, 2.7.3], the ideal I_n ($n \geq 1$) is primitive (Lemma 2(iv)). This means that there is a pure state ϕ_n of A such that $\phi_n(e_n) = 1$ and $\phi_n(I_n) = 0$. Let U be a free ultra-filter of the natural numbers and let ϕ be the state of A given by $\lim_U \phi_n(a) = \phi(a)$ for all $a \in A$. We show that the canonical representation π of A induced by ϕ is an irreducible representation with kernel I .

If ψ is a state of A , set $L(\psi) = \{a \in A | \psi(a^*a) = 0\}$. To show π is irreducible, it is sufficient to show that, given a and b in A with $\|a - L(\phi)\| = \|b - L(\phi)\| = 1$ and any $\varepsilon > 0$, there is a $c \in A$ with $\|c\| \leq 1 + \varepsilon$ and $ca - L(\phi) = b - L(\phi)$. Given $0 < \eta < 1$ with $(1 + \eta)(1 - \eta)^{-1} < 1 + \varepsilon$, there is a $u \in U$ so that $\|a - L(\phi_i)\|$ and $\|b - L(\phi_i)\|$ are in the real interval $[1 - \eta, 1 + \eta]$ for all $i \in u$. Since ϕ_i is a pure state, there is a $c_i \in A$ with $\|c_i\| \leq (1 + \eta)(1 - \eta)^{-1} \leq 1 + \varepsilon$ such that $c_i a - L(\phi_i) = b - L(\phi_i)$ for $i \in u$ [6]. Now let f_i be the range projection of $b e_i$ and let g_i be the range projection of $a e_i$. Since $e_i \leq e_{i+1}$, we have that $f_i \leq f_{i+1}$ and $g_i \leq g_{i+1}$ for every i . For convenience, let $f_0 = g_0 = 0$. The elements $c = \sum \{(f_i - f_{i-1})c_i(g_i - g_{i-1}) | i \in u\}$ exists in the strong operator topology and $\|c\| \leq 1 + \varepsilon$. For any $i \in u$ we get

$$\begin{aligned} ca - L(\phi_i) &= cae_i - L(\phi_i) = c(g_i - g_{i-1})a - L(\phi_i) \\ &= (f_i - f_{i-1})c_i(g_i - g_{i-1})a - L(\phi_i) \\ &= (f_i - f_{i-1})b - L(\phi_i) = b - L(\phi_i). \end{aligned}$$

Hence, we get $ca - L(\phi) = b - L(\phi)$. Consequently π is irreducible.

We have that $\pi(e_i) = 0$ for every i ; indeed, there is a $u \in U$ with $\{1, 2, \dots, i+1\} \cap u = \emptyset$. Since π certainly vanishes on $[\zeta]$, the kernel of π contains I (Lemma 3(iii)). Conversely, let $f \in (A) \setminus I$. We show $\pi(f) \neq 0$. Suppose we have found $p_i \in (Z) \setminus \zeta$, $f_i \in (fAf)$, and $v_i \in A$ for $1 \leq i \leq n$ such that the f_i are mutually orthogonal and such that $v_i^* v_i = (e_i - e_{i-1})p_i$ and $v_i v_i^* = f_i$ ($1 \leq i \leq n$). Here $e_0 = 0$. There is a maximal p_{n+1} in (Z) such that $(e_{n+1} - e_n)p_{n+1} < (f - \sum \{f_i | 1 \leq i \leq n\})p_{n+1}$. We have that $p_{n+1} \notin \zeta$; otherwise, $f \in I_{n+2}$, a contradiction. There is a partial isometry v_{n+1} in A such that $v_{n+1} v_{n+1}^* = f_{n+1} \leq f - \sum \{f_i | 1 \leq i \leq n\}$ and $v_{n+1}^* v_{n+1} = (e_{n+1} - e_n)p_{n+1}$. By

induction, we may assume p_i, f_i, v_i are defined for all $i=1, 2, \dots$. Setting $v = \sum v_i$ in the strong operator topology, we get that $\pi(f)v - L(\phi) = v - L(\phi) \neq 0$ since $\phi_i(v^*v) = \phi_i(p_{i+1}(e_{i+1} - e_i)v^*v(e_{i+1} - e_i)) = \phi_i(e_{i+1}) = 1$ for every i . This completes the proof for strictly increasing sequences.

The proof for decreasing sequences is analogous to proof for the increasing sequence. We also note that here a proof is also available from the theory of C^* -algebras using the fact that the space of primitive ideals is a Baire space with the hull-kernel topology (cf. [2, 3.4.13]). Q.E.D.

COROLLARY 5. *Let I be an ideal of A generated by the sequence $\{a_i\}$ in A and the ideal ζ .*

(1) *Then I satisfies one of the following conditions:*

(i) $I \in P$.

(ii) $I = J(e)$ for some $e \in (A)$.

(2) *If I contains a properly infinite projection not in ζ , then $I \in P$.*

(3) *If A is purely infinite, then $I \in P$.*

PROOF. (1) The ideal I is generated by ζ and the spectral projections e_n ($n=1, 2, \dots$) of $\sum a_i^*a_i/2^i(\|a_i\|^2 + 1)$ corresponding to the intervals $[n^{-1}, +\infty)$. Thus, we find that $I = \bigvee J(e_n)$. If $I \neq J(e_n)$ for all n , then $I \in P$ by Theorem 4.

(2) If e is a properly infinite projection in $I \setminus [\zeta]$, then $I = J(eve_n)$. Here, however, each $J(eve_n)$ is primitive (Lemma 3(iii) and (iv)).

(3) If $I = [\zeta]$, then I is primitive [4, Theorem 4.7]. The other case is treated in (2). Q.E.D.

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