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AN EXTREMAL PROPERTY OF SOME CAPACITARY MEASURES IN E_n[1]

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ABSTRACT. The capacitary measure on an arc of the circle is known (via conformal mapping) to be that measure of a class of measures which has the largest potential at certain points of the plane. Here it is shown that the analogous result is true in E_n.

1. **Introduction.** Let E_n be Euclidean n space, $n \geqq 3$; denote vectors $(x_1, \cdots, x_n) \in E_n$ by x; and let $|x|$ be the norm of x.

Put $S=\{x \in E_n : |x|=1\}$ and for a given positive c, $1 \leqq c < \infty$, let $H(c)$ denote the class of those positive measures μ on S of total mass 1 which satisfy

$$\phi(x) \leqq c \quad \text{for each } x \in E_n. \tag{1.1}$$

Here $\phi(x)=\int_S |x-y|^{2-n}\, d\mu(y)$ is the potential of μ.

Let $x \in E_n$, $|x| \neq 0$, $|x| \neq 1$. In this paper we characterize the measure in $H(c)$ which has the largest potential at x. The corresponding problem, where now ϕ is a logarithmic potential, occurs in E_2 in the study of the class S^* of starlike univalent functions f in $\{|x|<1\}$, normalized by $f(0)=0$, $f'(0)=1$, and satisfying $|f| \leqq e^{2c}$. Indeed, if ϕ is given, then the function f defined by $\log(|f(z)|/|z|)=2\phi(z)$, $|z|<1$, $f'(0)=1$, is in S^*. Conversely, given $f \in S^*$, $\frac{1}{2}\log(|f(z)|/|z|)$ is a potential whose associated measure is in $H(c)$. Hence the problem of characterizing the measure in $H(c)$ which has largest potential at x is equivalent to finding the function f in S^* whose modulus is largest at x. Due to this equivalence, in E_2 the measure can be characterized by using conformal mapping. The measure has the same form in E_n. That is

THEOREM 1. *Let $x \in E_n$, $|x| \neq 0$, $|x| \neq 1$, $n \geqq 3$. Let λ be such that the* cap $\{y : y \in S, |x-y| \leqq \lambda\}$ *has capacity $1/c$. Then if e is the capacitary measure on this cap, the measure ce has a greater potential at x than any other measure in $H(c)$.*

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With minor changes our method provides a new proof of the corresponding result in E_2.

We remark that T. J. Suffridge in E_2 [3] and J. L. Lewis in E_3 [2] have considered a related problem for potentials bounded below in $\{x:|x|\leqq 1\}$ by a constant.

2. **Several lemmas.** Given α, $0<\alpha\leqq\pi$, let $C_\alpha=\{x\in S:x_1>\cos\alpha\}$. Let ∂C_α denote $\{x\in S:x_1=\cos\alpha\}$ and $\bar{C}_\alpha=\partial C_\alpha\cup C_\alpha$. Let μ_α be the capacitary measure on $\bar{C}_\alpha$ and let P_α be its potential. Then

$$P_\alpha(x)=\int_{\bar{C}_\alpha}|x-y|^{2-n}\,d\mu_\alpha(y). \tag{2.1}$$

P_α is continuous on $E_n\cup\{\infty\}$, harmonic in $E_n-\bar{C}_\alpha$, 1 on $\bar{C}_\alpha$ and 0 at ∞, and is superharmonic in E_n. For fixed c, $1\leqq c<\infty$, let $\alpha_0=\alpha_0(c)$ be such that $\Phi=cP_{\alpha_0}$ satisfies $\Phi(0)=1$, that is the potential of $c\mu_{\alpha_0}$ at 0 is 1. The existence of α_0 is easily shown. We first prove

LEMMA 1. *If ϕ is the potential of a measure in $H(c)$ then for $0<\alpha\leqq\pi$*

$$\int_{\bar{C}_\alpha}\phi\,d\mu_\alpha\leqq\int_{\bar{C}_\alpha}\Phi\,d\mu_\alpha.$$

PROOF. The lemma is trivial if $\alpha\leqq\alpha_0$ since $\Phi=c$ on $\bar{C}_{\alpha_0}$. If $\alpha>\alpha_0$ we interchange the order of integration in $\int_{\bar{C}_\alpha}\phi\,d\mu_\alpha$ and obtain for ϕ with μ in $H(c)$ that

$$\int_{\bar{C}_\alpha}\phi\,d\mu_\alpha=\int_S P_\alpha(y)\,d\mu(y)\leqq\int_S d\mu(y)=1.$$

Similarly we have

$$\int_{\bar{C}_\alpha}\Phi\,d\mu_\alpha=\int_S cP_\alpha(y)\,d\mu_{\alpha_0}=\int_{\bar{C}_{\alpha_0}}c\,d\mu_{\alpha_0}=1.$$

Next we note that

(i) P_α has continuous one sided normal derivatives on C_α.

(ii) $\mu_\alpha(C_\theta)$ is a continuous function of θ, $0<\theta\leqq\pi$.

(i) is easily verified using the Kelvin transformation to map $\{x:|x|<1\}$ onto a halfspace (see Helms [1, pp. 36, 37]), and the reflection principle. (ii) follows from the fact that μ_α is capacitary measure on $\bar{C}_\alpha$. Let σ denote Lebesgue measure on S and σ_n the Lebesgue measure of S. Differentiating (2.1) under the integral sign we find for rx, $0<r<1$, $|x|=1$, that

$$r^{2-n/2}\frac{\partial}{\partial r}[r^{n/2-1}P_\alpha(rx)]=\left(\frac{n}{2}-1\right)\int_{\bar{C}_\alpha}\frac{1-r^2}{|rx-y|^n}\,d\mu_\alpha(y). \tag{2.2}$$

The last integral is a constant multiple, $(n/2-1)\sigma_n$, of the Poisson integral with boundary measure μ_α evaluated at rx. It follows from this fact that

$$(2.3)\qquad (n/2-1)^{-1}\sigma_n^{-1}\lim_{r\to 1} r^{2-n/2}(\partial/\partial r)[r^{n/2-1}P_\alpha(rx)] = d\mu_\alpha(x)/d\sigma$$

almost everywhere with respect to σ. Since (i) is true we deduce that μ_α is absolutely continuous with respect to σ on C_α and thereupon that equality holds in (2.3) whenever $x \in \partial C_\alpha$. Since we also have (ii) it follows that μ_α is absolutely continuous on $\bar{C}_\alpha$.

We observe from the boundary values of P_α and (2.3) that the values of $d\mu_\alpha(x)/d\sigma$ depend only on the x_1 coordinate of x. Hence if we define θ by $x_1=\cos\theta$, $0\leqq\theta<\alpha$, and put $h_\alpha(\theta)=d\mu_\alpha(x)/d\sigma$, $0\leqq\theta<\alpha$, then h is well defined. With this notation we prove

LEMMA 2. *If $0<\alpha\leqq\pi$, then h_α is a nonnegative nondecreasing function of θ on $[0, \alpha]$ which is not everywhere* 0.

PROOF. Let x_0 be that point on S with $x_1=1$, $x_i=0$ if $i\neq 1$. We first show for $0<r<1$ that

$$(2.4)\qquad P_\alpha(ry) \leqq P_\alpha(rx)$$

when $x, y \in C_\alpha$ and $|x-x_0|<|y-x_0|$. To prove (2.4) let K be the hyperplane through the origin for which $\tilde{x}=y$, where $\tilde{x}$ denotes the reflection of x with respect to K. Let G denote the halfspace bounded by K which contains x. Define f in $K\cup G$ by $f(z)=P_\alpha(\tilde{z})$, $z \in K\cup G$. Then $f-P_\alpha\leqq 0$ on $K\cup(C_\alpha\cap G)$. Since $f-P_\alpha$ is bounded and harmonic in $G-\bar{C}_\alpha$ it follows from the maximum principle that $f\leqq P_\alpha$ in G and consequently (2.4) is true.

Using (2.4) we get

$$\int_0^s \frac{\partial}{\partial r}[r^{n/2-1}P_\alpha(ry)]\,dr \leqq \int_0^s \frac{\partial}{\partial r}[r^{n/2-1}P_\alpha(rx)]\,dr$$

for $0<s\leqq 1$. Since equality holds for $s=1$ it follows that

$$\lim_{r\to 1}(\partial/\partial r)[r^{n/2-1}P_\alpha(rx)] \leqq \lim_{r\to 1}(\partial/\partial r)[r^{n/2-1}P_\alpha(ry)].$$

Using (2.3) we conclude that h_α is nondecreasing on $[0, \alpha)$. Finally, h_α is somewhere positive since $\mu_\alpha(C_\alpha)>0$.

3. **Proof of Theorem 1.** It clearly suffices to prove Theorem 1 for points x of the form $x=tx_0$, $0<t<\infty$, $t\neq 1$. Let Q be the Poisson kernel. Let $\phi \in H(c)$, $\phi\neq\Phi$, and suppose by way of contradiction that $\Phi(tx_0)\leqq\phi(tx_0)$. Put

$$(3.1)\qquad F(\theta) = \int_{C_\theta} Q(tx_0, y)[\Phi(y) - \phi(y)]\,d\sigma(y), \qquad 0 < \theta \leqq \pi.$$

Then $\Phi(tx_0)-\phi(tx_0)=F(\pi)$, and so the following lemma immediately gives Theorem 1.

LEMMA 3. *If* $\phi\neq\Phi$, *then* $F(\alpha)>0$, $\alpha_0\leqq\alpha\leqq\pi$.

PROOF. We observe $F(\alpha)\geqq 0$, $0<\alpha<\alpha_0$, and $F(\alpha_0)>0$, since $\Phi=c$ on $\bar{C}_{\alpha_0}$. Hence if Lemma 3 is false there exists δ, $\alpha_0<\delta\leqq\pi$, such that $F(\alpha)>0$, $\alpha_0\leqq\alpha<\delta$, and $F(\delta)=0$.

Define $\hat{Q}$ on $[0,\pi]$ by $\hat{Q}(\theta)=Q(tx_0,y)$ where y satisfies $y_1=\cos\theta$, and define $g_\delta(\theta)=h_\delta(\theta)/\hat{Q}(\theta)$. Since $\hat{Q}$ is strictly decreasing on $[0,\pi]$, Lemma 2 implies g_δ is nondecreasing and nonconstant on $[\alpha_0,\delta)$.

If $\alpha_0<\beta<\delta$ then

$$\int_{C_\beta}(\Phi-\phi)\,d\mu_\delta=\int_0^\beta g_\delta(\theta)\,dF(\theta)=F(\beta)g_\delta(\beta)-\int_0^\beta F(\theta)\,dg_\delta(\theta). \tag{3.2}$$

The left-hand integral approaches

$$\int_{C_\delta}(\Phi-\phi)\,d\mu_\delta \quad \text{as } \beta\to\delta,$$

since μ_δ is absolutely continuous with respect to σ. Also

$$\int_\beta^\delta g_\delta(\theta)\,|dF(\theta)|\geqq F(\beta)g_\delta(\beta)$$

and so $\lim_{\beta\to\delta}F(\beta)g_\delta(\beta)=0$. Taking the limit as $\beta\to\delta$ in (3.2), we get

$$\int_{C_\delta}(\Phi-\phi)\,d\mu_\delta=-\int_0^\delta F(\theta)\,dg_\delta(\theta).$$

The right-hand side is negative since g_δ is nondecreasing and nonconstant on $[0,\delta]$ and F is positive on $[\alpha_0,\delta)$. However by Lemma 1 the left-hand integral is nonnegative. We have reached a contradiction. Hence Lemma 3 is true.

4. **Remark.** We note that Φ also solves the corresponding minimum problem. That is

$$\Phi(-tx_0)<\min_{|x|=t}\phi(x), \tag{4.1}$$

when ϕ is a potential of a measure on $H(c)$, ϕ is not a rotation of Φ, $0<t<\infty$, and $t\neq 1$. The proof is exactly the same except we substitute $-Q(-tx_0,y)$ for $Q(tx_0,y)$ in (3.1). It appears likely that (4.1) is also true for $t=1$, although we have not been able to prove this.

References

1. L. L. Helms, *Introduction to potential theory*, Pure and Appl. Math., vol. 22, Interscience, New York, 1969. MR **41** #5638.

2. J. L. Lewis, *A potential theory problem in three space* (to appear).

3. T. J. Suffridge, *A coefficient problem for a class of univalent functions*, Michigan Math. J. **16** (1969), 33–42. MR **39** #1646.

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