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ERGODIC PROPERTIES OF BOUNDED L_1-OPERATORS

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ABSTRACT. Individual ergodic theorems for bounded L_1-operators are proved in §1, and the problem of existence of positive invariant functions for positive L_1-operators is considered in §2. A decomposition theorem similar to that of Sucheston [12] is proved in the last section.

1. **Individual ergodic theorems.** Let $(X, \mathscr{M}, m)$ be a σ-finite measure space and $L_p(X)=L_p(X, \mathscr{M}, m)$, $1\leqq p\leqq\infty$, the usual (complex) Banach spaces. If $A \in \mathscr{M}$ then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X-A$. Let T be a bounded linear operator on $L_1(X)$ and τ its linear modulus [2]. Thus τ is a positive linear operator on $L_1(X)$ such that $\|\tau\|_1=\|T\|_1$ and $\tau g=\sup\{|Tf|; f\in L_1(X)$ and $|f|\leqq g\}$ for any $0\leqq g\in L_1(X)$. The adjoint of T is denoted by T^*. Throughout this section it will be assumed that there exists a strictly positive function s in $L_\infty(X)$ such that

$$\tau^* s \leqq s \quad \text{a.e.} \tag{1}$$

Clearly if T is a contraction then $\tau^*1\leqq 1$ a.e. Let $a_{n,k}$ $(n, k=0, 1, \cdots)$ be a matrix of numbers such that

$$\lim_n \sum_{k=0}^{\infty} a_{n,k} = 1, \tag{2}$$

$$\lim_{n'} \sum_{k=0}^{\infty} a_{n',k} b_{k+1} = b \tag{3}$$

whenever $b_0, b_1, \cdots$ is a bounded sequence of numbers for which

$$\lim_{n'} \sum_{k=0}^{\infty} a_{n',k} b_k = b$$

exists and is finite, where (n') is a subsequence of (n). Let $w_1, w_2, \cdots$ be a sequence of nonnegative numbers whose sum is one, and let $u_0, u_1, \cdots$ be the sequence defined by $u_0=1$ and $u_n=w_n u_0+\cdots+w_1 u_{n-1}$ for $n\geqq 1$. In this section, under these conditions, we shall prove the following theorems.

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THEOREM 1. *If $p_0, p_1, \cdots$ is a sequence of nonnegative measurable functions on X with $|Tg| \leqq P_{n+1}$ a.e. whenever $g \in L_1(X)$ and $|g| \leqq p_n$ a.e. then for any $f \in L_1(X)$ the limit*

$$(*) \qquad \lim_n \left(\sum_{k=0}^{n} u_k T^k f(x) \right) \Big/ \left(\sum_{k=0}^{n} u_k p_k(x) \right)$$

exists and is finite a.e. on the set $\{x \in X;\ \sum_{k=0}^{\infty} u_k p_k(x) > 0\}$.

THEOREM 2. *Suppose there exists a strictly positive function h in $L_1(X)$ such that*

(i) $\sum_{k=0}^{\infty} a_{n,k} \tau^k h$ exists in the weak topology for any n, and

(ii) the set $\{\sum_{k=0}^{\infty} a_{n,k} \tau^k h;\ n \geqq 0\}$ is weakly sequentially compact in $L_1(X)$.

Then for any $f \in L_1(X)$ the limit

$$(**) \qquad \lim_n \left(\sum_{k=0}^{n} u_k T^k f(x) \right) \Big/ \left(\sum_{k=0}^{n} u_k \right)$$

exists and is finite a.e.

PROOF OF THEOREM 1. For $sf \in L_1(X)$, where $f \in L_1(X)$, define

$$(4) \qquad V_T(sf) = sTf \quad \text{and} \quad V_\tau(sf) = s\tau f.$$

Since $\{sf; f \in L_1(X)\}$ is a dense subspace of $L_1(X)$ and $\|V_T(sf)\|_1 \leqq \|V_\tau(s|f|)\|_1 = \|s\tau|f|\,\|_1 = \|(\tau^* s)|f|\,\|_1 \leqq \|sf\|_1$, V_T and V_τ may be considered to be linear contractions on $L_1(X)$. An easy argument shows that V_τ coincides with the linear modulus of V_T. Let $g \in L_1(X)$ and $|g| \leqq sp_n$ a.e., and choose an increasing sequence $g_1, g_2, \cdots$ of nonnegative integrable functions on X such that $\lim_n sg_n = |g|$ a.e. Then $|V_T g| \leqq V_\tau |g| = \lim_n s\tau g_n \leqq sp_{n+1}$ a.e., and hence the ergodic theorem of **[9]** completes the proof of Theorem 1.

PROOF OF THEOREM 2. Let $g \in L_1(X)$ and (n') a subsequence of (n) such that $\sum_{k=0}^{\infty} a_{n',k}\, \tau^k h$ converges weakly to g. Then it follows from a slight modification of an argument of **[10]** that

$$(5) \qquad \lim_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} V_\tau^k(sh) - sg \right\|_1 = 0,$$

and that $sg > 0$ a.e. on C and $sg = 0$ a.e. on D, where C and D denote the conservative and dissipative parts **[1]** of V_τ, respectively. Hence Theorem 1 completes the proof of Theorem 2.

2. **Invariant functions.** In this section we shall assume that $(X, \mathcal{M}, m)$ is a probability space and T is a positive linear operator on $L_1(X)$ such that there exists a strictly positive function s in $L_\infty(X)$ with $T^* s \leqq s$ a.e.

The operator T is called conservative if $\sum_{k=0}^{\infty} T^k f(x)=\infty$ a.e. for any strictly positive function $f \in L_1(X)$. A measurable set A is called closed if $f \in L_1(A)$ implies $Tf \in L_1(A)$. The purpose of this section is to prove the following theorems.

THEOREM 3. *If T is conservative and satisfies the condition of Theorem 2 then there exists a strictly positive function g in $L_1(X)$ with $Tg=g$ and hence if, in addition,*

$$\sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \right\|_1 < \infty \tag{6}$$

then the mean ergodic theorem holds for T; i.e., for any $f \in L_1(X)$ the sequence $(1/n) \sum_{k=0}^{n-1} T^k f$ converges in the norm topology.

COROLLARY 1 (CF. FONG [5, THEOREM 3]). *If T satisfies (6), then a necessary and sufficient condition that T have a strictly positive invariant function in $L_1(X)$ is that T be conservative and for any $A \in \mathcal{M}$ the limit*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \int T^{*k} 1_A \, dm \tag{7}$$

exist.

THEOREM 4. *If T satisfies (6), then the following conditions are equivalent.*
(0) *There exists a strictly positive function $f_0 \in L_1(X)$ with $Tf_0=f_0$.*
(i) *$A \in \mathcal{M}$ and $m(A)>0$ imply* $\inf_n \int T^{*n} 1_A \, dm>0$.
(ii) *$A \in \mathcal{M}$ and $m(A)>0$ imply*

$$\lim_n \left(\sup_j \frac{1}{n} \sum_{k=0}^{n-1} \int T^{*k+j} 1_A \, dm \right) > 0.$$

THEOREM 5. *If T satisfies (6), then the space X is the disjoint union of two uniquely determined measurable sets P and N such that*
(a) *P is closed,*
(b) *there exists an $h \in L_1(P)$ with $h>0$ a.e. on P and $Th=h$,*
(c) *for any $f \in L_1(X)$ the limit*

$$\tilde{f}(x) = \lim_n \frac{1}{n} 1_P \sum_{k=0}^{n-1} T^k f(x)$$

exists and is finite a.e., $s\tilde{f} \in L_1(P)$ and $T\tilde{f}=\tilde{f}$ a.e.; moreover we have

$$\lim_n \left\| s\left(\frac{1}{n} 1_P \sum_{k=0}^{n-1} T^k f - \tilde{f} \right) \right\|_1 = 0,$$

(d) *if $N=X-P$ then N is a union of countably many sets $A_i \in \mathscr{M}$ with*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \int_{A_i} T^k f\, dm = 0$$

for any $0 \leqq f \in L_1(X)$.

Theorem 4 is a generalization of results obtained by Neveu [7] (see also [8]), Dean and Sucheston [3] and Fong [5], and Theorem 5 is a generalization of results obtained by Krengel [6] and Fong [5].

PROOF OF THEOREM 3. The first half of the theorem is direct from the argument in the proof of Theorem 2, and the second half follows from the mean ergodic theorem (cf. [4, Theorem VIII.5.1]).

PROOF OF THEOREM 4. For the purpose of proof we introduce a third condition:

(iii) *$A \in \mathscr{M}$ and $m(A)>0$ imply*

$$\liminf_n \frac{1}{n} \sum_{k=0}^{n-1} \int T^{*k} 1_A\, dm > 0.$$

The proof follows the scheme (0)$\Rightarrow$(ii)$\Rightarrow$(i)$\Rightarrow$(iii)$\Rightarrow$(0). The implication (i)$\Rightarrow$(iii) is obvious. The following two implications (0)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) follow from the same arguments as in [5, p. 80]. Thus we prove here only the implication (iii)$\Rightarrow$(0).

Let L be a Banach limit and define a positive linear functional φ on $L_\infty(X)$ by the relation

$$\varphi(u) = L\left(\frac{1}{n} \sum_{k=0}^{n-1} \int T^{*k} u\, dm\right), \qquad u \in L_\infty(X).$$

If we denote by T^{**} the adjoint of T^* then, for any $0 \leqq u \in L_\infty(X)$, $(T^{**}\varphi-\varphi)u=\varphi(T^*u-u)=L((1/n)\int (T^{*n}u-u)\,dm)\geqq L((1/n)\int T^{*n}u\,dm)\geqq 0$, and hence $T^{**}\varphi-\varphi\geqq 0$. Thus if we let $\varphi_n=(1/n)\sum_{k=0}^{n-1} T^{**k}\varphi$ then $0\leqq\varphi\leqq\varphi_1\leqq\varphi_2\leqq\cdots$ and the $\|\varphi_n\|$ are bounded, whence there exists a positive linear functional φ_∞ on $L_\infty(X)$ such that $\lim_n\|\varphi_\infty-\varphi_n\|=0$. It is now easy to see that $T^{**}\varphi_\infty=\varphi_\infty$. Set $\mu(A)=\varphi_\infty(1_A)$ for $A \in \mathscr{M}$. Then μ is a finitely additive measure on $\mathscr{M}$ vanishing on sets of m measure zero. Let $\mu=\mu_m+\mu_c$ be the unique decomposition of μ, where $\mu_m\geqq 0$ is a countably additive measure on $\mathscr{M}$ and where $\mu_c\geqq 0$ is a finitely additive measure on $\mathscr{M}$ such that if $\lambda\geqq 0$ is a countably additive measure on $\mathscr{M}$ with $\lambda\leqq\mu_c$ then $\lambda=0$ [11, Theorem 3]. Then $T^{**}\mu_m\leqq T^{**}\mu=\mu=\mu_m+\mu_c$ and hence $T^{**}\mu_m-\mu_m\leqq\mu_c$, from which it follows easily that $T^{**}\mu_m\leqq\mu_m$. We next show that μ_m is equivalent to m. Assume the contrary: there exists a set $E \in \mathscr{M}$ with $\mu_m(E)=0$ and $m(E)>0$. Then Theorem 4 of [11] implies that there exists a set $A \in \mathscr{M}$ with $A\subset E$, $\mu_c(A)=\mu(A)=0$ and

$m(A)>0$. But this is impossible because, by (iii),

$$\mu(A) \geqq \varphi_\infty(1_A) \geqq \varphi(1_A) = L\left(\frac{1}{n}\sum_{k=0}^{n-1} T^{*k}1_A\, dm\right)$$

$$\geqq \liminf_n \frac{1}{n}\sum_{k=0}^{n-1}\int T^{*k}1_A\, dm > 0.$$

To complete the proof it is now sufficient to show that $T^{**}\mu_m=\mu_m$. But to see this, since $T^{**}\mu_m \leqq \mu_m$, it suffices to show that $T^*s=s$. If this fails to hold then there exists a set $A\in\mathcal{M}$ with $m(A)>0$ and a positive constant c such that $s-T^*s\geqq c1_A$, and hence we have

$$\liminf_n \frac{1}{n}\sum_{k=0}^{n-1}\int cT^{*k}1_A\, dm \leqq \liminf_n \frac{1}{n}\sum_{k=0}^{n-1}\int T^{*k}(s-T^*s)\, dm$$

$$= \liminf_n \frac{1}{n}\int (s-T^*s)\, dm$$

$$\leqq \lim_n \frac{1}{n}\int s\, dm = 0,$$

which contradicts (iii). This completes the proof of Theorem 4.

Proof of Theorem 5. An argument similar to that of [**5**, Proposition 2] is sufficient, and we omit the details.

Remark 1. It may be readily seen that if T has a strictly positive invariant function $f_0\in L_1(X)$ then the class $\mathcal{I}$ of all closed sets forms a σ-subfield of $\mathcal{M}$ (cf. [**1**]). Thus if $f\in L_1(X)$, we shall denote by $E\{f|\mathcal{I}\}$ the conditional expectation of f with respect to $\mathcal{I}$. Applying the Chacon identification theorem [**1**], we have the following results.

(a) *If $f\in L_1(X)$ then*

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^k f = f_0\frac{E\{sf\mid\mathcal{I}\}}{E\{sf_0\mid\mathcal{I}\}} \quad a.e.$$

(b) *If $u\in L_\infty(X)$ then*

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^{*k}u = s\frac{E\{uf_0\mid\mathcal{I}\}}{E\{sf_0\mid\mathcal{I}\}} \quad a.e.$$

3. **Decomposition theorem.** In this section we shall prove the following

Theorem 6 (cf. Sucheston [**12**, Theorem 1]). *If T is a positive linear operator on $L_1(X)$ satisfying* (6), *then the space X uniquely decomposes into two measurable sets Y and Z such that*

(i) *$f\in L_1(Z)$ implies $Tf\in L_1(Z)$,*

(ii) *if $f\in L_1(Z)$ then* $\lim_n\|(1/n)\sum_{k=0}^{n-1} T^k f\|_1=0$,

(iii) *there exists a nonnegative function s in* $L_\infty(Y)$ *with* $s>0$ *a.e. on* Y *and* $T^*s=s$.

PROOF. If we let $u=\limsup_n(1/n)\sum_{k=0}^{n-1}T^{*k}1$, then an easy calculation shows that $u\in L_\infty(X)$ and $T^*u\geqq u$ a.e. Next if we let $s=\lim_n(1/n)\times\sum_{k=0}^{n-1}T^{*k}u$, then it follows that $0\leqq s\in L_\infty(X)$ and $T^*s=s$. Put $Y=\{x\in X;\, s(x)>0\}$ and $Z=X-Y$. If $0\leqq f\in L_1(Z)$ then

$$\lim_n\int\left(\frac{1}{n}\sum_{k=0}^{n-1}T^kf\right)dm=\lim_n\int f\left(\frac{1}{n}\sum_{k=0}^{n-1}T^{*k}1\right)dm$$
$$\leqq\int fu\,dm\leqq\int fs\,dm=0.$$

Thus (ii) follows. (i) is clear. The proof is complete.

REMARK 2. By Theorem 1, if $f\in L_1(X)$ and $0\leqq g\in L_1(X)$ then the limit

$$\lim_n\left(\sum_{k=0}^{n}T^kf(x)\right)\bigg/\left(\sum_{k=0}^{n}T^kg(x)\right)$$

exists and is finite a.e. on $Y\cap\{x\in X;\,\sum_{k=0}^{\infty}T^kg(x)>0\}$. But in general this does not hold on $Z\cap\{x\in X;\,\sum_{k=0}^{\infty}T^kg(x)>0\}$ (see Fong [**5**, p. 77]).

COROLLARY 2. *Let* T *be a positive linear operator on* $L_1(X)$ *satisfying* (6), *and suppose that* $\limsup_n\|(1/n)\sum_{k=0}^{n-1}T^kg\|_1>0$ *for any* $0\leqq g\in L_1(X)$ *with* $\|g\|_1>0$. *Then there exists a strictly positive function s in* $L_\infty(X)$ *with* $T^*s=s$.

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