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ON GOING DOWN FOR SIMPLE OVERRINGS

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ABSTRACT. Let R be an integral domain with quotient field K. If R is Noetherian: then the Krull dimension of R is at most $1 \Leftrightarrow$ for all overrings S of R, $R \subset S$ satisfies going down. R is Dedekind (resp., PID) $\Leftrightarrow R$ is Krull (resp., UFD) and, for all $u \in K$, $R \subset R[u]$ satisfies going down. R is Prüfer $\Leftrightarrow R$ is integrally closed, every intersection of two principal ideals of R is finitely generated, and $R \subset R[u]$ satisfies going down for all $u \in K$.

1. **Introduction and notation.** Let R be a (commutative integral) domain with integral closure $\bar{R}$ and quotient field K. Our main purpose is to study going down (GD) between R and its overrings (that is, R-subalgebras of K). Using GD, we obtain characterizations of Prüfer domains in Corollary 4, of Noetherian domains of (Krull) dimension at most 1 in Proposition 7 and Corollary 9, and of Dedekind domains and PID's in Corollary 10. As may be expected from the characterization of Bézout domains given by Dawson and the author in [**1**, Corollary 4.4], a special role is played by simple overrings (that is, ones generated over R or $\bar{R}$ by single elements of K).

Any unexplained terminology is standard, as in [**2**], [**3**], and [**7**].

2. **Domains characterized via going down.** We begin by quoting the following result about FC domains (i.e., domains for which every intersection of two principal ideals is finitely generated). Observe that a domain is FC if and only if every element of its quotient field has finitely generated conductor.

LEMMA 1 (MCADAM [**6**, THEOREM 2]). *Let R be FC and let T be an overring of R such that R is integrally closed in T and $R \subset T$ satisfies lying over (LO). Then $R=T$.*

LEMMA 2. *Let R be quasi-local. Assume either*
(a) *R is a Krull domain such that* $\dim(R) \geqq 2$ *or*
(b) *R is an integrally closed FC domain which is not a valuation ring.*

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Then there exists $u \in K$ such that $u \notin R$, $u^{-1} \notin R$ and $R \subset R[u]$ does not satisfy GD.

PROOF. If R is a Krull domain, the assumption about its dimension readily implies that R is not a valuation ring (cf. [**2**, Theorem 35.16]). Hence, in case either (a) or (b) holds, there exists $u \in K$ such that $u \notin R$ and $u^{-1} \notin R$. If M is the maximal ideal of R, then Chevalley's lemma [**3**, Theorem 55] implies that M survives in either $R[u]$ or $R[u^{-1}]$. Without loss of generality, $MR[u] \neq R[u]$, and so there exists a prime of $R[u]$ lying over M.

Now, suppose that the lemma is false. Then $R \subset R[u]$ satisfies GD and, since R is quasi-local, $R \subset R[u]$ also satisfies LO. In case (b), Lemma 1 shows $u \in R$, a contradiction. For case (a), let P be any prime of R of height 1. As R_P is a discrete (rank 1) valuation ring and $R_P \subset R_P[u] = R[u]_{R \setminus P}$ inherits LO from $R \subset R[u]$, Lemma 1 implies $u \in R_P$. Hence $u \in \bigcap R_P = R$, a contradiction, to complete the proof.

THEOREM 3. *Assume either* (a) *R is a Krull domain such that* $\dim(R) \geqq 2$ *or* (b) *R is an integrally closed FC domain which is not Prüfer. Then there exists $u \in K$ such that $R \subset R[u]$ does not satisfy GD.*

PROOF. (a) Let M be a maximal ideal of R of height at least 2. Note R_M is Krull [**2**, Corollary 35.6], quasi-local, of dimension at least 2. If the result is false, then $R_M \subset R_M[u] = R[u]_{R \setminus M}$ inherits GD from $R \subset R[u]$, for all $u \in K$. This contradicts case (a) of Lemma 2.

(b) Since R is not Prüfer, there exists a maximal ideal M of R such that R_M is not a valuation ring. As R_M is integrally closed and FC, the proof concludes as above, this time using case (b) of Lemma 2.

COROLLARY 4. *R is Prüfer if and only if the following three conditions hold*: (i) *R is integrally closed*; (ii) *R is FC*; (iii) *$R \subset R[u]$ satisfies GD for all $u \in K$.*

PROOF. Let R be Prüfer. It is well known that (i) and (ii) hold. As explained prior to [**1**, Proposition 3.6], (iii) also holds.

The converse follows immediately from case (b) of Theorem 3.

We remark that Corollary 4 extends a result of Quentel [**8**, Corollaire 2]. A different extension appears in [**6**, Theorem 1].

Lemma 5 (resp., Proposition 8) will, in special cases, relate the question of unibranchedness of R in $\bar{R}$ to that of GD for simple overrings generated by elements of $\bar{R}$ (resp., of $K \setminus \bar{R}$). Lemma 5 was essentially proved by McAdam in [**5**, Theorem 2].

LEMMA 5. *Assume R is Noetherian, T is an overring of R, and $R \subset R[u]$ satisfies GD for all $u \in T$. If P is a prime of R of height at least* 2, *then at most one prime of T lies over P.*

COROLLARY 6. *Let R be Noetherian and T an overring of R contained in $\bar{R}$. The following conditions are equivalent*:

(i) *$R\subset T$ satisfies GD*;

(ii) *$R\subset S$ satisfies GD for all rings $R\subset S\subset T$*;

(iii) *$R\subset R[u]$ satisfies GD for all $u\in T$.*

PROOF. For rings $R\subset S\subset T$, integrality shows $S\subset T$ satisfies LO, whence [**5**, Lemma 1(2)] yields (i)$\Rightarrow$(ii). Clearly, (ii)$\Rightarrow$(iii). Finally, assume (iii). If $R\subset T$ fails to satisfy GD, there exist primes $P_1\subsetneq P_2$ of R and Q of T such that $Q\cap R=P_2$ and no prime of T contained in Q lies over P_1. Since $R\subset T$ satisfies GU and LO [**3**, Theorem 44] and P_2 is unibranched in T (by Lemma 5), we easily obtain a contradiction. Thus (iii)$\Rightarrow$(i), to complete the proof.

PROPOSITION 7. *Let R be Noetherian. The following conditions are equivalent*:

(i) $\dim(R)\leqq 1$;

(ii) *$S\subset T$ satisfies GD for all overrings $S\subset T$ of R*;

(iii) *$R\subset S$ satisfies GD for all overrings S of R.*

PROOF. (i)$\Rightarrow$(ii) follows immediately from the Krull-Akizuki theorem [**3**, Theorem 93], and (ii)$\Rightarrow$(iii) is clear. Finally, if P is a prime of R of height greater than 1, then [**4**, §13] supplies a discrete (rank one) valuation overring V of R such that the maximal ideal of V lies over P. Since $R\subset V$ does not satisfy GD, we conclude that (iii)$\Rightarrow$(i).

PROPOSITION 8. *Assume $R\subset \bar{R}[u]$ satisfies GD for all $u\in K\backslash\bar{R}$. Assume either* (a) *$\bar{R}$ is a Krull domain such that* $\dim(R)\geqq 2$ *or* (b) *$\bar{R}$ is FC and not Prüfer. Then there exists a nonmaximal prime of R which is not unibranched in $\bar{R}$.*

PROOF. Note that $\dim(\bar{R})=\dim(R)$. Hence, by applying the appropriate case of Theorem 3, there exists $u\in K$ such that $\bar{R}\subset\bar{R}[u]$ does not satisfy GD. Thus, there exist primes $M_1\subsetneq M_2$ of $\bar{R}$ and Q of $\bar{R}[u]$ such that $Q\cap\bar{R}=M_2$ and no prime of $\bar{R}[u]$ contained in Q lies over M_1. Let $P_i=M_i\cap R$ $(i=1, 2)$. Since $R\subset\bar{R}[u]$ satisfies GD, there exists a prime N of $\bar{R}[u]$ such that $N\subset Q$ and $N\cap R=P_1$. Then P_1 is not unibranched in $\bar{R}$, as $N\cap\bar{R}$ and M_1 are distinct. Of course, the INC property for integral extensions [**3**, Theorem 44] shows $P_1\neq P_2$, and so P_1 is nonmaximal, to complete the proof.

We next show that, under an assumption of unibranchedness, we may restrict consideration in part (iii) of Proposition 7 to simple overrings S.

COROLLARY 9. *Let R be Noetherian such that every prime of R of height* 1 *is unibranched in $\bar{R}$. Then* $\dim(R)\leqq 1$ *if and only if $R\subset\bar{R}[u]$ satisfies GD for all $u\in K\backslash\bar{R}$.*

PROOF. The "only if" assertion is trivial. If the "if" assertion fails, a consequence of the principal ideal theorem [**3**, Theorem 152] allows us to choose a prime M of R of height 2. As Mori's theorem [**7**, Theorem 33.10] shows $\bar{R}$ is Krull, it follows that $A=(\bar{R})_{R\setminus M}$, the integral closure of R_M, is also Krull. If $u \in K\setminus A$, then $R_M \subset A[u]$ inherits GD from $R \subset \bar{R}[u]$. Thus, case (a) of Proposition 8 provides a nonmaximal prime P of R_M which is not unibranched in A. Then $P \cap R$ is a prime of R of height 1 which is not unibranched in $\bar{R}$, a contradiction, to complete the proof.

COROLLARY 10. *Let R be Krull (resp., UFD). Then R is Dedekind (resp., PID) if and only if $R \subset R[u]$ satisfies GD for all $u \in K$.*

PROOF. As explained prior to [**1**, Proposition 3.6], the "only if" assertions are immediate. To obtain the converses, apply [**2**, Propositions 31.6 and 35.2 and Theorem 35.16] and case (a) of Theorem 3.

REMARKS. (i) The use of Theorem 3 in the preceding proof characterizing PID's may be replaced by an appeal to [**1**, Corollary 4.4], since any UFD is a GCD.

(ii) Since Lemma 1 appears in [**7**, (33.1), p. 114] for the case of Noetherian R, the proofs of Lemma 2(a), Theorem 3(a), Propositions 7 and 8(a), and Corollaries 9 and 10 do not depend on the results in [**6**].

(iii) The question of characterizing the domains satisfying GD with simple overrings is far from settled. We close by sketching an example[2] of a quasi-local integrally closed two-dimensional domain R such that $R \subset R[u]$ satisfies GD for all $u \in K$ and R is not valuation.

First, let S be a quasi-local integrally closed one-dimensional domain which is not valuation (for example, construct S as in [**3**, 2-1, Exercise 5]). Let M be the maximal ideal of S and F the quotient field of S. Define R to be the restricted power series ring $S+xF[[x]]$. Then R is not valuation, is integrally closed in its quotient field $K=F((x))$, and has only two nonzero primes, viz., $M+xF[[x]]$ and $P=xF[[x]]$.

It remains to show $R \subset R[u]$ satisfies GD whenever $u \in K$. One reduces quickly to the case $u \notin R$, $u^{-1} \notin R$. If $u \notin F[[x]]$, then $u^{-1} \in P$, a contradiction. Hence $u=v+w$ for some $v \in F\setminus S$ and $w \in P$. Since $R[u]=R[v]=S[v]+P$, we see that P is also a prime of $R[u]$, whence $R \subset R[u]$ satisfies GD.

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