DETERMINANTAL VARIETIES, MONOMIAL SEMIGROUPS, AND ALGEBRAS ASSOCIATED WITH IDEALS¹

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ABSTRACT. This paper is concerned with the Rees and symmetric algebras of powers of ideals generated by A -sequences. These algebras are represented as quotients of polynomial rings over A by ideals defined by minors of matrices. Their Krull dimensions are computed when A is a finite domain over a field. When A is a polynomial ring over a Cohen-Macaulay ring A_0 and the A -sequence consists of indeterminates, the Rees algebra is shown to be Cohen-Macaulay. If furthermore A_0 is a finite domain over a field, the symmetric algebra is shown to be Cohen-Macaulay only for A -sequences and squares of A -sequences of length two. Connections with algebras generated by monomials and Veronese varieties are pointed out.

1. Introduction. Let A be a commutative ring with unit and $\mathbf{b} = (b_1, \dots, b_r)$ an ideal of A . Denote by $S(\mathbf{b})$ the symmetric algebra of $\mathbf{b}$ and by $R(\mathbf{b})$ the Rees algebra of $\mathbf{b}$. Thus $R(\mathbf{b}) = A[b_1T, \dots, b_rT] \subseteq A[T]$. These graded algebras can be represented as quotients of polynomial rings via $\varphi: A[X_1, \dots, X_r] \rightarrow S(\mathbf{b})$ given by $\varphi(X_i) = b_i$ and $\gamma: A[X_1, \dots, X_r] \rightarrow R(\mathbf{b})$ given by $\gamma(X_i) = b_iT$. Setting $Q = \text{Ker } \varphi$ and $Q_\infty = \text{Ker } \gamma$, we have that Q is generated by all linear forms $\sum c_i X_i$ such that $\sum c_i b_i = 0$ and Q_∞ is generated by all forms $F(X_1, \dots, X_r)$ such that $F(b_1, \dots, b_r) = 0$. We are interested in this paper in the case where $\mathbf{b} = \mathbf{a}^n$, $\mathbf{a} = (a_1, \dots, a_s)$, $a_1, \dots, a_s$ an A -sequence, $n \geq 1$. In this situation, Q and Q_∞ are ideals defined by minors of matrices whose heights we compute when A is a finite integral domain over a field. In the case where A is a polynomial ring over a Cohen-Macaulay ring A_0 and $a_1, \dots, a_s$ a subset of the variables, it is shown that $R(\mathbf{a}^n)$ is always Cohen-Macaulay whereas $S(\mathbf{a}^n)$ is Cohen-Macaulay only when $s=1$ or $n=1$ or $s=n=2$. The proof of the first employs Hochster's notion of a normal semigroup of monomials.

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For the symmetric algebra we assume also that A_0 is a finite integral domain over a field. In the final section, we exhibit some connections between these graded algebras and Veronese varieties, reestablishing some properties of these varieties due to Gröbner.

2. Determinantal ideals and dimension of $R(\mathbf{a}^n)$ and $S(\mathbf{a}^n)$. Let $a_1, \dots, a_s$ be an A -sequence, $\mathbf{a} = (a_1, \dots, a_s)$, $n \geq 1$. We want to compute the Krull dimension of $R(\mathbf{a}^n)$ and $S(\mathbf{a}^n)$ and determine when these rings are Cohen-Macaulay. The starting point for our considerations is the following theorem stating that Q and Q_∞ are determinantal ideals.

THEOREM 1. *Let $a_1, \dots, a_s$ be an A -sequence, $\mathbf{a} = (a_1, \dots, a_s)$, $n \geq 1$. Set $B = A[X_I]$ where $X_I = \{X_{i_1, \dots, i_s} \mid \sum i_a = n, i_a \geq 0\}$, a set of indeterminates of cardinality $\binom{n+s-1}{n}$. Define a $[1 + \binom{n+s-2}{n-1}] \times s$ matrix U , the bottom $\binom{n+s-2}{n-1}$ rows of which are indexed by $J = \{(j_1, \dots, j_s) \mid \sum j_a = n-1, j_a \geq 0\}$ as follows: $U_{1,k} = a_k$ and $U_{(j_1, \dots, j_s), k} = X_{j_1, \dots, j_{k-1}, j_{k+1}, j_{k+1}, \dots, j_s}$. Then $R(\mathbf{a}^n) \approx B/Q_\infty$ and $S(\mathbf{a}^n) \approx B/Q$ where Q_∞ is the ideal generated by all 2×2 minors of U and Q is the ideal generated by those 2×2 minors of U involving the first row.*

PROOF. [1, Theorems 1 and 4].

COROLLARY 1. *If $s=1$, then $R(\mathbf{a}^n) \approx S(\mathbf{a}^n) \approx A[X]$. Thus $\dim R(\mathbf{a}^n) = \dim S(\mathbf{a}^n) = 1 + \dim A$ and each is Cohen-Macaulay if A is Cohen-Macaulay.*

COROLLARY 2. *If $n=1$, then $R(\mathbf{a}) \approx S(\mathbf{a})$. If A is Cohen-Macaulay, then each is Cohen-Macaulay of dimension $1 + \dim A$.*

PROOF. U is the $2 \times s$ matrix with first row $(a_1, \dots, a_s)$ and second row $(X_1, \dots, X_s)$. Thus $Q = Q_\infty$ and $R(\mathbf{a}) \approx S(\mathbf{a})$. It is not hard to show [1, Proposition 2] that Q is a perfect ideal of grade $s-1$. Thus $S(\mathbf{a})$ is Cohen-Macaulay if A is and $\dim S(\mathbf{a}) = \dim B - \text{ht } Q = s + \dim A - \text{gr } Q = 1 + \dim A$.

We can assume henceforth that $s, n \geq 2$.

PROPOSITION 1. *Notation as in Theorem 1. Then $Q_\infty \cdot \mathbf{a}B \subseteq Q \subseteq Q_\infty \cap \mathbf{a}B$.*

PROOF. Let $F = U_{(j),k} U_{(j'),k'} - U_{(j'),k} U_{(j),k'}$ be a generator of Q_∞ not among the generators of Q . We must show that $a_m F \in Q$ for $m=1, \dots, s$. Consider the matrix

$$\begin{pmatrix} a_k & a_{k'} & a_m \\ U_{(j),k} & U_{(j),k'} & U_{(j),m} \\ U_{(j'),k} & U_{(j'),k'} & U_{(j'),m} \end{pmatrix}.$$

Comparing the expressions for the determinant, first by expanding along

the third row, then by expanding along the third column, we see that $a_m F \in Q$. Thus $Q_\infty \cdot aB \subseteq Q$. The other inequality is obvious.

COROLLARY. $\text{ht } Q = \min(\text{ht } Q_\infty, s)$.

PROOF. Since a is generated by an A -sequence of length s , $\text{ht } aB = s$. In general for any ideals I and J in a Noetherian ring, $\text{ht}(I \cap J) = \text{ht}(IJ) = \min(\text{ht } I, \text{ht } J)$.

Thus if we know $\text{ht } Q_\infty$, we know $\text{ht } Q$. Furthermore we can compute $\text{ht } Q_\infty$ from $\dim R(a^n)$ and $\dim B$ provided that B satisfies the chain condition $\text{ht } b + \dim b = \dim B$ for any ideal b of B .

PROPOSITION 2. Suppose that A is a finite integral domain over a field k , $0 \neq b$ any ideal of A . Then $\dim R(b) = 1 + \dim A$.

PROOF. Let K be the fraction field of A and $b = (b_1, \dots, b_r)$. Then $R(b) = A[b_1 T, \dots, b_r T] \subseteq A[T]$. The fraction field of $R(b)$ contains K and $T = b_1^{-1}(b_1 T)$ and so equals $K(T)$. Thus $\dim R(b) = \text{td}_k(K(T)) = 1 + \text{td}_k(K) = 1 + \dim A$.

COROLLARY 1. Notation as in Theorem 1 with A a finite integral domain over a field. Then $\text{ht } Q_\infty = \binom{n+s-1}{n} - 1$.

PROOF. $\dim B = \dim R(a^n) + \text{ht } Q_\infty$ where $\dim B = \dim A + \binom{n+s-1}{n}$ and $\dim R(a^n) = \dim A + 1$.

COROLLARY 2. Notation as in Theorem 1 with A a finite integral domain over a field. Then (i) $\text{ht } Q = s$, (ii) $\dim S(a^n) = \binom{n+s-1}{n} - s + \dim A$.

PROOF. (i) By the corollary to Proposition 1, it suffices to observe that $\binom{n+s-1}{n} \geq s+1$ when $s, n \geq 2$. In fact,

$$(s-1)! \binom{n+s-1}{n} \geq \frac{(s+1)!}{2} \geq (s-1)!(s+1)$$

where the first inequality is strict unless $n=2$ and the second inequality is strict unless $s=2$.

$$(ii) \quad \dim S(a^n) = \dim B - \text{ht } Q = \binom{n+s-1}{n} + \dim A - s.$$

3. Monomial semigroups and Cohen-Macaulay properties. We now turn to the question of when $R(a^n)$ or $S(a^n)$ is Cohen-Macaulay, considering only the special case $A = A_0[T_1, \dots, T_s, T_{s+1}, \dots, T_r]$, $a = (T_1, \dots, T_s)$, A_0 a ring.

PROPOSITION 3. Let $A = A_0[T_1, \dots, T_s, T_{s+1}, \dots, T_r]$, $a = (T_1, \dots, T_s)$, $R(a^n) \approx B/Q_\infty$, $S(a^n) \approx B/Q$ where $B = A[X_i]$ is a polynomial ring over A in $\binom{n+s-1}{n}$ indeterminates. Then $Q = Q_\infty \cap aB$.

PROOF. It suffices by Proposition 1 to show that $Q_\infty \cap \mathfrak{a}B \subseteq Q$. Set $B = B_0[T_1, \dots, T_s]$ where $B_0 = A_0[T_{s+1}, \dots, T_r, X_I]$. By Theorem 1, $Q_\infty = (b_1, \dots, b_p, c_1, \dots, c_q)$ where $b_i \in B_0$ and $(c_1, \dots, c_q) = Q \subseteq \mathfrak{a}B$. If $x \in Q_\infty \cap \mathfrak{a}B$, then $x = \sum f_i b_i + y$ where $f_i \in B$, $y \in Q \subseteq \mathfrak{a}B$.

Writing $f_i = g_i + h_i$ where $g_i \in B_0$, $h_i \in \mathfrak{a}B$, we have $x = \sum g_i b_i + \sum h_i b_i + y$ where $\sum h_i b_i \in Q_\infty \cdot \mathfrak{a}B \subseteq Q$ by Proposition 1. Thus

$$\sum g_i b_i \in B_0 \cap \mathfrak{a}B = (0)$$

and so $x \in Q$.

THEOREM 2. *Notation as in Proposition 3. Suppose that A_0 is a finite integral domain over a field and Cohen-Macaulay. Then $S(\mathfrak{a}^n)$ is Cohen-Macaulay if and only if $s=n=2$.*

PROOF. If $S(\mathfrak{a}^n)$ is Cohen-Macaulay, then Q must be an unmixed ideal of B . The previous proposition shows that Q is the intersection of two prime ideals whose heights are different unless $s=n=2$ (Corollary 2, Proposition 2).

Conversely, if $s=n=2$, then $B = A[X_1, X_2, X_3]$ and $Q = (T_1 X_2 - T_2 X_1, T_1 X_3 - T_2 X_2)$. The generators of Q clearly form a B -sequence. Hence Q is a perfect ideal. Since B is Cohen-Macaulay, it follows that $S((T_1, T_2)^2)$ is Cohen-Macaulay.

The notion of a normal semigroup of monomials introduced by Hochster turns out to be very useful in looking at the Rees algebra. The next proposition and theorem give equivalent definitions of this concept and the key property of adjoining such a semigroup to a Cohen-Macaulay ring.

A monomial in the indeterminates $X_1, \dots, X_n$ is a term of the form $X_1^{m_1} X_2^{m_2} \dots X_n^{m_n}$ where $m_i \geq 0$. A semigroup of monomials M is a set of monomials closed under product and containing the monomial $1 = X_1^0 \dots X_n^0$. If R is any ring over which $X_1, \dots, X_n$ are algebraically independent, we can consider the semigroup ring $R[M] \subseteq R[X_1, \dots, X_n]$. An integral domain is called normal if it is Noetherian and integrally closed.

PROPOSITION 4. *Let M be a monomial semigroup in $X_1, \dots, X_n$. The following conditions are equivalent:*

- (i) M is finitely generated and for any $p, q, r \in M$ and any integer $m \geq 1$ satisfying $pq^m = r^m$, there exists $t \in M$ such that $p = t^m$.
- (ii) There exists a field K such that $K[M] \subseteq K[X_1, \dots, X_n]$ is normal.
- (iii) M is finitely generated and for any integrally closed domain D , $D[M] \sim [X_1, \dots, X_n]$ is integrally closed.

PROOF. [6, Proposition 1].

DEFINITION. A monomial semigroup satisfying the conditions of Proposition 4 is called normal.

EXAMPLE. Let $M_{s,n}$ be the semigroup generated by all monomials of degree n in $X_1, \dots, X_s$. Then $M_{s,n}$ is finitely generated and is easily seen to satisfy condition (i) by taking $t=rq^{-1}$. It is in fact what Hochster calls a full ($\Rightarrow$ normal) semigroup.

THEOREM 3. *If R is a Cohen-Macaulay ring and M is a normal monomial semigroup, then $R[M]$ is Cohen-Macaulay.*

PROOF. [6, Theorem 1].

In applying these ideas, we need the following fact whose proof can be found in [1, Corollary 2, Theorem 5].

PROPOSITION 5. *Let A_0 be an integrally closed domain,*

$$A = A_0[T_1, \dots, T_s, T_{s+1}, \dots, T_r],$$

$a = (T_1, \dots, T_s)$, $n \geq 1$. Then $R(a^n)$ is integrally closed.

Combining the last three results, we obtain

THEOREM 4. *Let A_0 be a Cohen-Macaulay ring, $A = A_0[T_1, \dots, T_s, T_{s+1}, \dots, T_r]$, $a = (T_1, \dots, T_s)$, $n \geq 1$. Then $R(a^n)$ is Cohen-Macaulay.*

PROOF. Let M be the monomial semigroup generated by $T_1, \dots, T_r$ and all monomials mT where m is a monomial of degree n in $T_1, \dots, T_s$. Then $R(a^n) \approx A_0[M] \subseteq A_0[T_1, \dots, T_r, T]$. By Proposition 5, $A_0[M]$ is an integrally closed domain if A_0 is an integrally closed domain. Since M is finitely generated, M is normal by Proposition 4. Then by Theorem 3, $R(a^n)$ is Cohen-Macaulay.

4. Veronese varieties and generic, symmetric matrices.

DEFINITION. The Veronese variety $V_{s,n}$ over a field k is the projective variety whose homogeneous coordinate ring is generated over k by all monomials of degree n in the indeterminates $T_1, \dots, T_s$, or, in our previous notation, $k[M_{s,n}]$.

Introducing $\binom{n+s-1}{n}$ indeterminates $X_I = \{X_{i_1 \dots i_s} \mid \sum i_\alpha = n, i_\alpha \geq 0\}$, let $I_{s,n}$ denote the kernel of the map $k[X_I] \rightarrow k[M_{s,n}]$ given by $X_{i_1 \dots i_s} \mapsto T_1^{i_1} \dots T_s^{i_s}$. $I_{s,n}$ is generated by all forms $F(X_I)$ such that $F(T^I) = 0$. In fact $I_{s,n}$ is generated by quadratic forms. More precisely, if we denote by U' the matrix obtained from the matrix U of Theorem 1 by removing the first row, then $I_{s,n}$ is generated by all 2×2 minors of U' (see [3], [4]). The normality of the monomial semigroup $M_{s,n}$ thus gives

THEOREM 5. *The coordinate ring of the Veronese variety $V_{s,n}$ is arithmetically normal and Cohen-Macaulay.*

REMARK. The "Cohen-Macaulay" part of the above theorem was proved by Gröbner [5] who showed by direct methods that $I_{s,n}$ is perfect.

Taking $n=2$, the matrix U' becomes a generic, symmetric $s \times s$ matrix. Thus we have the following

COROLLARY. Let X be a generic, symmetric $s \times s$ matrix, that is, the matrix whose (i, j) and (j, i) entries are X_{ij} ($1 \leq i \leq j \leq s$). Denote by I the ideal generated by all $t \times t$ minors of X in the polynomial ring $k[X_{ij}]$. Then I_t is perfect of grade $\binom{s+2-t}{2}$ for $t=1, 2$, or s .

PROOF. The result is trivial for $t=1$ or s . For $t=2$, I_2 is just the defining ideal of $V_{s,2}$ which is Cohen-Macaulay. Hence I_2 is perfect. Furthermore

$$\text{gr } I_2 = \text{ht } I_2 = \dim k[X_{ij}] - \dim k[M_{s,2}] = \binom{s+1}{2} - s = \binom{s}{2}.$$

Note that $\dim k[M_{s,2}] = s$ since $k[M_{s,2}]$ is an integral extension of $k[X_1^2, \dots, X_s^2]$.

It seems reasonable to conjecture that the above corollary is true for all values of t .² It is also interesting to compare this fact with the recent theorem of Hochster and Eagon ([7], [8]) that the grade of the ideal generated by $t \times t$ minors of an $r \times s$ matrix is bounded above by $(r-t+1)(s-t+1)$ and if equality holds, the ideal is perfect. This was previously known for $t=\min(r, s)$ [2] and $t=2$ ([12], [13]). For the generic, symmetric $s \times s$ matrix, I_2 is perfect of grade $\binom{s}{2} < (r-t+1) \times (s-t+1) = (s-1)^2$ for $s \geq 3$. Perhaps the corresponding theorem holds for symmetric $s \times s$ matrices with the upper bound $\binom{s+2-t}{2}$.

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² The referee has informed me that this conjecture has been settled by R. Kutz [14].

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