THE DILATATION OF HARMONIC QUASI- CONFORMAL FUNCTIONS

WILLIAM V. CALDWELL

ABSTRACT. In this paper, it is shown that the dilation of a harmonic quasiconformal function is a subharmonic function.

1. Introduction. Let D be an open simply-connected domain in the complex plane C . A real-valued function φ is said to be in class C^r in D if the partial derivatives of φ of order $k \leq r$ exist and are continuous in D . If $f = u + iv$ is defined in D , we will say $f \in C^r$ if u and v are in class C^r . All functions considered in this paper will be assumed to be in C^r where $r \geq 3$ and the *Jacobian* matrix of f will be denoted by J_f . It is well known that a sufficient condition for $f: D \rightarrow C$ to be orientation-preserving and locally 1:1 is that $\det J_{f(z)}$ be positive in D . Let $\det J_{f(z)} > 0$ in D and $z \in D$. Then for any angle θ , we define

$$D_\theta f(z_0) = \lim_{r \rightarrow 0} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}}.$$

Note that $e^{i\theta} D_\theta f(z_0)$ is just the covariant derivative of f at z_0 in the direction given by θ and cannot vanish if $\det J_{f(z_0)} \neq 0$. We define the *dilatation* of f at z_0 by

$$D_{f(z_0)} = \frac{\max_\theta |D_\theta f(z_0)|}{\min_\theta |D_\theta f(z_0)|}.$$

Note that $D_{f(z)} \geq 1$ for all $z \in D$ and if $D_{f(z)} = 1$ in D , then f is holomorphic. If f is 1:1 in D , f is said to be quasiconformal and if $D_{f(z)} \leq k$ in D , then f is said to be k -quasiconformal in D . The purpose of this paper is to show that if f is a harmonic quasiconformal function in D , then D_f is subharmonic in D and D_f is harmonic if and only if $D_{f(z)}$ is constant. The proof will be achieved by examining the coefficients of an elliptic system of first order partial differential equations determined by f .

2. Elliptic systems and differentiable quasiconformal functions. Every differentiable quasiconformal function determines two distinct elliptic

Received by the editors October 27, 1972.

AMS (MOS) subject classifications (1970). Primary 30A60, 30A93.

Key words and phrases. Quasiconformal, dilatation.

systems, each of which may be regarded as a generalization of the *Cauchy-Riemann* equations. These systems may be obtained in the following way:

If f is differentiable and quasiconformal in D , the first complex dilatation of f at a point $z \in D$ is given by $u = f_z/f_z$ and the second complex dilatation of f is $v = \bar{f}_z/\bar{f}_z$.

For the first case, one considers the set W of functions defined and differentiable in D such that if $g \in W$, $g_z = \mu g_z$. This system is called a *Beltrami* system and it is well known that W is an algebra over the field of complex numbers and that if $g \in W$, then g is locally quasiconformal in D except for at most a countable set E having no limit point in D . (Recall that we have assumed that $f \in C^r$ where $r \geq 3$.)

We will be considering the second case. Let W be the set of functions defined and differentiable in D and such that if $g \in W$, then $g_z = \nu \bar{g}_z$. This system was studied extensively by *L. Bers* who called elements of W "pseudo-analytic functions of the second kind" [2]. As in the first case, if $g \in W$, then g is locally quasiconformal in D except for at most a countable subset E having no limit point in D . A tedious but elementary computation shows that ν can be written in the form

$$\nu = (\alpha^2 + \beta^2 - \alpha^{-2} + 2i\beta\alpha^{-1})(\alpha^2 + \beta^2 + \alpha^{-2} + 2)^{-1}$$

where α and β are real-valued functions and $\alpha \neq 0$ in D . Both systems have a matrix representation [3]. The matrix representation of the *Bers* system is of the form

$$J_f = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$$

where $f^* = p + iq$ is a complex-valued function satisfying a system of the form $f_z^* = Af_z^* + B\bar{f}_z^*$ [3]. *Bers* called functions satisfying a system of this kind "pseudo-analytic functions of the first kind." It is not difficult to show that $\bar{\nu}$ and $\alpha\beta + i\alpha^2$ are holomorphic if and only if f is harmonic. Furthermore, if $\alpha\beta + i\alpha^2$ (hence $\bar{\nu}$) is holomorphic, then $\alpha^{-1}f^* = \alpha^{-1}(p + iq)$ is also holomorphic and for $f^* = p + iq$ such that $\alpha^{-1}f^*$ is holomorphic in D , f^* determines a harmonic function f , which is unique up to an additive constant, such that $f_z = \bar{\nu}f_z$.

3. The dilatation of f . At this point, it is simpler to abandon complex notation temporarily. We consider a domain D in E^2 and a differentiable mapping $f: D \rightarrow E^2$ such that

$$J_f = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} p & -q \\ q & p \end{pmatrix}, \quad p^2 + q^2 \neq 0.$$

We denote the transpose of J_f by J_f^t . Then at a point (x_0, y_0) in D , $J_f^t J_f$ and

$J_f J_f^t$ are positive definite symmetric matrices with common eigenvalues

$$\lambda_1^2 = 2^{-1}(p^2 + q^2)[\alpha^2 + \beta^2 + \alpha^{-2} + ((\alpha^2 + \beta^2 + \alpha^{-2})^2 - 4)^{1/2}]$$

and

$$\lambda_2^2 = 2^{-1}(p^2 + q^2)[\alpha^2 + \beta^2 + \alpha^{-2} - ((\alpha^2 + \beta^2 + \alpha^{-2})^2 - 4)^{1/2}].$$

If u_1 and u_2 are the corresponding orthogonal eigenvectors of $J_f J_f^t$ and if $\lambda_i = \sqrt{\lambda_i^2}$ ($i=1, 2$), then $D_{u_i} f(x_0, y_0) = \lambda_i v_i$ ($i=1, 2$), where v_1 and v_2 are orthogonal eigenvectors of $J_f J_f^t$ [4]. (It is easy to show that in this case v_1 and v_2 depend only on the values of α and β at (x_0, y_0) .) If we assume $\lambda_2 \leq \lambda_1$, the dilatation of f at (x_0, y_0) is given by

$$\begin{aligned} D_f &= \lambda_1 \lambda_2^{-1} \\ &= [\alpha^2 + \beta^2 + \alpha^{-2} + ((\alpha^2 + \beta^2 + \alpha^{-2})^2 - 4)^{1/2}]^{1/2} \\ &\quad \times [\alpha^2 + \beta^2 + \alpha^{-2} + ((\alpha^2 + \beta^2 + \alpha^{-2})^2 - 4)^{1/2}]^{-1/2} \\ &= 2^{-1}[\alpha^2 + \beta^2 + \alpha^{-2} + ((\alpha^2 + \beta^2 + \alpha^{-2})^2 - 4)^{1/2}]. \end{aligned}$$

We will say that f has an umbilical point at (x_0, y_0) if $D_f = 1$ at that point.

Note that if W is the set of differentiable functions defined in D and satisfying the Bers system with coefficients α and β , then for $g \in W$, D_g is determined except at a critical point and $D_g = 1$ at a point z_0 if and only if $\alpha(z_0) = 1$ and $\beta(z_0) = 0$. For brevity, we will denote the dilatation of an element of W by D .

THEOREM. *Let D be a domain in C , let $f: D \rightarrow C$ be harmonic and quasi-conformal in D , and let D be the dilatation of f . Then D is subharmonic in D and is harmonic if and only if D is constant.*

PROOF. Let α and β be the coefficients of the Bers system determined by f and let $M = \alpha^2 + \beta^2 + \alpha^{-2}$. Then $D = 2^{-1}(M + (M^2 - 4)^{1/2})$. Note that if f is holomorphic, then $M = 2$, $D = 1$, and the theorem holds. We suppose $M \neq 2$. An easy computation shows that

$$\Delta D = D\{\Delta M(M^2 - 4)^{-1/2} + (M^2 - 4)^{-3/2}((M^2 - 4)^{1/2}(M_x^2 + M_y^2))\}.$$

Since $\alpha\beta + i\alpha^2$ is holomorphic, we obtain that $\beta_x = 2\alpha_y - \beta\alpha^{-1}\alpha_x$ and $\beta_y = -2\alpha_x - \beta\alpha^{-1}\alpha_y$. It follows that $M_x^2 + M_y^2 = 4\alpha^{-2}(M^2 - 4)(\alpha_x^2 + \alpha_y^2)$ and $\Delta M = 8\alpha^{-2}M(\alpha_x^2 + \alpha_y^2)$. Substituting,

$$\begin{aligned} \Delta D &= 4D\alpha^{-2}(M^2 - 4)^{-1/2}\{[2M + (M^2 - 4)^{1/2} - M](\alpha_x^2 + \alpha_y^2)\} \\ &= 2D^2\alpha^{-2}(M^2 - 4)^{-1/2}(\alpha_x^2 + \alpha_y^2) \geq 0. \end{aligned}$$

Therefore D is subharmonic and is harmonic if and only if α , hence β , is constant. But if α and β are constant, so is D .

Note that $D_x = D(M^2 - 4)^{-1/2}M_x$ and $D_y = D(M^2 - 4)^{-1/2}M_y$. An easy

computation shows that

$$D_x^2 + D_y^2 = D^2[M^2 - 4]^{-1}(M_x^2 + M_y^2) = 4D^2\alpha^{-2}(\alpha_x^2 + \alpha_y^2).$$

It is easy to verify that if $\alpha\beta + i\alpha^2$ is univalent in D and if f has no umbilical points in D , then for every open set $U \subset D$, D_f has neither a maximum nor a minimum in R . If, however, f has an umbilical point at z_0 , D has an absolute minimum at z_0 but need not be differentiable at that point.

REFERENCES

1. L. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand Math. Studies, no. 10, Van Nostrand, Princeton, N.J., 1966. MR 34 #336.
2. L. Bers, *Theory of pseudo-analytic functions*, New York University, 1963. (Mimeographed lecture notes).
3. W. V. Caldwell, *Some relationships between Bers and Beltrami systems and linear elliptic systems of partial differential equations*, Canad. J. Math. 17 (1965), 627–642. MR 34 #496.
4. C. Lanczos, *Linear systems in self-adjoint form*, Amer. Math. Monthly 65 (1958), 665–679. MR 21 #1978.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, FLINT, MICHIGAN 48503