ON PRÜFER RINGS AS IMAGES OF PRÜFER DOMAINS

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ABSTRACT. Only commutative rings with unity will be considered in this paper. It is shown that if R is the homomorphic image of a Prüfer domain, then R is a Prüfer ring but that the converse is not true in general. It is then shown that a Prüfer ring R is the homomorphic image of a Prüfer domain if and only if the total quotient ring of R is the homomorphic image of a Prüfer domain. A class of total quotient rings which satisfy this last condition is then presented.

Only commutative rings with unity will be considered in this paper. An element of a ring is said to be a *regular* element in case it is not a zero divisor and an ideal is said to be *regular* in case it contains a regular element. Griffin [2] defined a ring to be Prüfer if every finitely generated regular ideal is invertible. Griffin shows that R is a Prüfer ring if and only if $A(B \cap C) = AB \cap AC$ for all ideals A, B, C of R , with B or C regular. Prüfer rings have been studied in [1], [2], [3], and an account of Prüfer rings can be found in [4].

First we will give a necessary condition for a ring to be a homomorphic image of a Prüfer domain. Then we will give an example of a Prüfer ring which does not satisfy the condition, thus providing a negative answer to a question asked informally by Robert Gilmer, namely that Prüfer rings are not necessarily images of Prüfer domains.

THEOREM 1. *If R is a homomorphic image of a Prüfer domain D , then $A(B \cap C) = AB \cap AC$ for any ideals A, B , and C of R . In particular, R is a Prüfer ring.*

PROOF. We always have $A(B \cap C) \subseteq AB \cap AC$. We may assume that A, B , and C are nonzero, for the contrary case presents no problem. Let ϕ be the homomorphism from D onto R and let A', B', C' be the preimages under ϕ of A, B , and C respectively. Let $x \in AB \cap AC$, say $x = \sum a_i b_i = \sum d_j c_j$ for $a_i, d_j \in A, b_i \in B, c_j \in C$. Let $a'_i, d'_j \in A', b'_i \in B',$ and $c'_j \in C'$ be any preimages of a_i, d_j, b_i, c_j respectively. Let $x' = \sum a'_i b'_i$. Then $\phi(x') = x$ and since $\phi(\sum d'_j c'_j) = x$, we have $x' = \sum a'_i b'_i = \sum d'_j c'_j + k'$ where $k' \in K'$, the kernel of ϕ . Hence $x' \in A' B' \cap (A' C' + K')$. But since D is a Prüfer domain,

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$A'B' \cap (A'C' + K') = (A'B' \cap A'C') + (A'B' \cap K') = A'(B' \cap C') + (A'B' \cap K')$. Hence $x = \phi(x') \in A(B \cap C)$. Consequently $A(B \cap C) = AB \cap AC$. $\square$

Now, for the example, let F be a field of characteristic not equal to two and let X and Y be indeterminates over F . Let $R = F[X, Y]/(X^2, Y^2)$ and make the obvious identification of images. We note that $a + bX + cY + dXY$ is regular if and only if $a \neq 0$ and in this case,

$$a^{-1} + (-ba^{-2})X + (-ca^{-2})Y + (2cba^{-3} - da^{-2})XY$$

is its inverse. Thus R is its own total quotient ring and therefore a Prüfer ring. Let $A = XR$, $B = (X - Y)R$ and $C = (X + Y)R$. Clearly $AB \cap AC = (XY)R$. If $z \in B \cap C$, then

$$\begin{aligned} z &= (a_1 + b_1X + c_1Y + d_1XY)(X - Y) \\ &= (a_2 + b_2X + c_2Y + d_2XY)(X + Y) \end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in F$. Hence

$$a_1X - a_1Y + c_1XY - b_1XY = a_2X + a_2Y + c_2XY + b_2XY.$$

Therefore $a_1 = a_2$ and $a_1 = -a_2$ which implies that $a_1 = a_2 = 0$ since F does not have characteristic two. Consequently, $z \in XYR$ and $A(B \cap C) = A((XY)R) = (0)$ which shows that $AB \cap AC \neq A(B \cap C)$. By Theorem 1, R is a Prüfer ring which is not a homomorphic image of a Prüfer domain.

Now that we have established the fact that not all Prüfer rings can be realized as homomorphic images of Prüfer domains, we turn to the natural question of which ones can be so realized. While this question will not be completely answered here, we will show that it is sufficient to settle the question for total quotient rings. If A is an ideal of R , denote the fractional ideal $R:A$ by A^{-1} . Then A is invertible if and only if $AA^{-1} = R$.

THEOREM 2. *A Prüfer ring is the homomorphic image of a Prüfer domain if and only if its total quotient ring is the homomorphic image of a Prüfer domain.*

PROOF. Let R be a Prüfer ring with total quotient ring T . Suppose that R is the homomorphic image of the Prüfer domain D under the homomorphism θ . Then let $S = \{s \in D \mid \theta(s) \text{ is a regular element of } R\}$ and let ϕ be the extension of θ to D_S obtained by $\phi(d/s) = \theta(d)/\theta(s)$. Then it is easy to see that ϕ is a homomorphism from the Prüfer domain D_S onto T .

Now let us assume that T is the homomorphic image of the Prüfer domain K under the homomorphism ϕ , say. Let $U = \{k \in K \mid \phi(k) \text{ is a unit in } T\}$. Let ψ be defined from K_U to T by $\psi(k/u) = \phi(k)\phi(u)^{-1}$. One can easily show that ψ is a homomorphism and K_U is clearly a Prüfer domain. Hence, without loss of generality, we may assume that $\phi(k)$ is a unit in T if and only if k is a unit in K . Let $D = \{k \in K \mid \phi(k) \in R\}$ and let θ be the

restriction of the mapping ϕ to D . Let $S = \{d \in D \mid \phi(d) \text{ is a regular element of } R\}$. Since $\phi(s)$ is a unit in T , $s^{-1} \in K$ and hence $D_S \subseteq K$. Let $k \in K$. Then $\phi(k) = \theta(d)/\theta(s)$ for some $d \in D$, $s \in S$. Therefore, $k = (d/s) - t$ where t belongs to the kernel of ϕ which is the kernel of θ , and thus t is in D . Hence, $k = (d - ts)/s \in D_S$. Therefore $K = D_S$.

Let $A = (a_1, a_2, \dots, a_n)$ be an ideal of D such that $\theta(A)$ is a regular ideal of R . Then there exist elements $t_1, t_2, \dots, t_n \in \theta(A)^{-1}$ such that $\sum_{i=1}^n t_i \theta(a_i) = 1$. Let $e_i \in K$ be such that $\phi(e_i) = t_i$, $i = 1, \dots, n$. Then $e_i \in A^{-1}$ for all i since $\phi(e_i a_i) = \phi(e_i) \phi(a_i) = t_i \theta(a_i) \in R$ and hence $e_i a_i \in D$ which implies that $e_i A \subseteq D$ and consequently $e_i \in A^{-1}$. Since $\phi(\sum e_i a_i) = 1$ is a unit in R , $\sum e_i a_i$ is a unit in K and its inverse is an element of D . Therefore, $\sum e_i a_i$ is a unit in D and hence AA^{-1} contains a unit which implies that A is invertible. Now let $a, b \in D$ such that $\theta(a)$ and $\theta(b)$ are zero divisors. Since a and b generate an invertible ideal in K , there exist elements $c', d' \in ((a, b)K)^{-1}$ such that $ac' + bd' = 1$. Since $ac', ad', bc', bd' \in K$, there exists an element $s \in S$ such that $sc', sd' \in ((a, b)D)^{-1}$, and if $sc' = c$ and $sd' = d$, then $ac + bd = s$. Therefore $B = (a, b, ac, ad, bc, bd)$ is a finitely generated ideal of D such that $\theta(B)$ is a regular ideal of R . Hence, by the case considered above, B is invertible and so there exist elements $e, f, g, h, i, j \in B^{-1}$ such that

$$ae + bf + acg + adh + bci + bdj = 1.$$

But clearly $e, f, cg, dh, ci, dj \in ((a, b)D)^{-1}$ and hence $(a, b)D$ is an invertible ideal of D . Therefore D is a Prüfer domain. $\square$

We are now in a position to describe a class of Prüfer rings which are homomorphic images of Prüfer domains. Let L be a field of characteristic n and let K be the direct sum of m copies of L where $m \leq n$ if $n \neq 0$ and m is unrestricted if $n = 0$. With this notation in hand we can prove the following.

THEOREM 3. *Let R be a Prüfer ring with total quotient ring K where K is as defined in the above paragraph. Then R is the homomorphic image of a Prüfer domain.*

PROOF. Let $M_i = (X - i)L[X]_{(X - i)}$, $i = 1, \dots, m$. (Here i is the identity of L added to itself i times.) Let $V_i = L + M_i$ for each $i = 1, \dots, m$. Let $f(X)/g(X)$ be an element of the quotient field of V_i . Then we can write $f(X)/g(X) = (f'(X)/g'(X))(X - i)^t$ where neither $f'(X)$ nor $g'(X)$ is divisible by $(X - i)$. If $t \geq 0$, $f(X)/g(X) \in V_i$ and if $t < 0$, $g(X)/f(X) \in V_i$; hence V_i is a valuation ring with maximal ideal M_i . Therefore, $V = \bigcap_{i=1}^m V_i$ is a Prüfer domain with maximal ideals $M_1V, \dots, M_mV$. Let $N = \bigcap_{i=1}^m M_iV$ and set $\bar{V} = V/N$. Now, in $\bar{V}$, the ideals $\bar{M}_1V, \dots, \bar{M}_mV$ form a pairwise comaximal set. Hence $\bar{V} = R_1 \oplus R_2 \oplus \dots \oplus R_m$ where $R_i \cong \bar{V}/\bar{M}_iV$, $i = 1, \dots, m$. But $\bar{V}/\bar{M}_iV \cong V/M_iV$ and we note that M_iV is precisely the kernel of the

homomorphism from V to L defined by replacing the indeterminate X with i . Hence $R_i \cong V/M_i V \cong L$. Therefore $K \cong R_1 \oplus \cdots \oplus R_m$ and so K is the homomorphic image of the Prüfer domain V .

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