NORMAL SUBGROUPS OF GROUPS WHICH ARE PRODUCTS OF TWO ABELIAN SUBGROUPS

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ABSTRACT. It is shown that if a group $G=AB$, where A and B are Abelian subgroups of G , $A \neq B$, and either A or B satisfies the maximum condition, then there is a normal subgroup N of G , $N \neq G$, such that N contains either A or B .

Summary and background. The purpose of this paper is to show that, if a group G is a nontrivial product of two Abelian subgroups A and B , and if either A or B satisfies the maximum condition, then there is a proper normal subgroup of G containing either A or B . (L. E. Knop [7]).

The basic impetus for this research is found in a paper of N. Itô [1]. In this paper, Itô showed that if G is a product of two Abelian subgroups A and B , then G is metabelian. Furthermore, he showed that if G is finite, then there is a normal subgroup of G , not the identity, which is contained in either A or B , and also that there is a proper normal subgroup of G which contains either A or B .

Additional work on the problem of when there is a normal subgroup contained in one of the two factors has been done by P. M. Cohn [2], who extended Itô's result to the case where A and B are infinite cyclic, and N. F. Sesekin [3], who extended the result to the case where A and B are finitely connected. E. Schenkman [4] claimed that both of Itô's results could be extended to the case where both A and B satisfied the maximum condition. Unfortunately there was a flaw in the proof that either A or B contained a normal subgroup of G (discovered by both Schenkman, Zentralblatt review [4], and W. R. Scott [5]), and the proof that either A or B was contained in a proper normal subgroup of G was based on this result. B. Amberg and W. R. Scott [6, Theorem 4] did show that if either A or B

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satisfies the maximum condition and if any one of three other conditions is satisfied, or, if both A and B are cyclic, then there is a proper normal subgroup containing one of the factors. This paper shows that additional conditions are not necessary, and establishes half of Schenkman's claim in a more general setting.

It should also be noted that recently B. Amberg [8] has announced that both of Schenkman's results hold in the more general setting where only one of the two factors is assumed to satisfy the maximum condition, and that additional facts about the structure of $G=AB$ where both A and B are Abelian subgroups satisfying the maximum condition can be established.

Notation, definitions, and lemmas. $\langle H \rangle$ is the subgroup generated by the set H . $(g, h) = g^{-1}h^{-1}gh$. $G' = \langle (g, h) \mid g \text{ and } h \text{ are in } G \rangle$ is the commutator subgroup of G . G is metabelian provided G' is Abelian. $C(D) = \{x \mid xd = dx \text{ for all } d \in D\}$ is the centralizer in G of the set D . $Z = C(G)$ is the center of G .

A group A satisfies the maximum condition provided any ascending chain of subgroups is finite. For Abelian groups, this is equivalent to the condition that A is finitely generated.

For a finitely generated Abelian group A , let $R(A)$ be the torsion free rank of A and let $T(A)$ be the order of the torsion subgroup of A .

Let $G=AB$ be a product of Abelian subgroups and let N be a normal subgroup of G . Define $A[N] = \{a \in A \mid ab \in N \text{ for some } b \in B\}$ and $B[N] = \{b \in B \mid ab \in N \text{ for some } a \in A\}$.

The following elementary result will be used often enough to deserve explicit mention.

LEMMA 1. *If C , D , and K are subgroups of the group $H=CD$, and $C \subseteq K$, then $K=C(D \cap K)$.*

The following result of Itô is, of course, basic to this work.

LEMMA 2 (ITÔ [1, THEOREM 1]). *Each (finite or infinite) group, which may be represented as the product of two Abelian subgroups, is metabelian.*

The next result can essentially be found in both Sesekin [3] and Schenkman [4].

LEMMA 3. *If $G=AB$ is a product of Abelian subgroups, and N is a normal subgroup of G , then $NA[N]=NB[N]=A[N]B[N]$ is a normal subgroup of G .*

PROOF. $A[N]=A \cap NB$ so $A[N]$ is a subgroup. N is normal in G so $NA[N]$ is also a subgroup. For any $a_0 \in A[N]$, $b \in B[N]$, there is an element

$a \in A[N]$ such that $ab \in N$. Thus $a_0b = (a_0a^{-1})(ab) \in A[N]N$. On the other hand, $N \subseteq A[N]B[N]$ by definition. Thus $NA[N] = A[N]B[N]$. Similarly, $NB[N] = A[N]B[N]$.

A is Abelian so A normalizes $NA[N]$. B normalizes $NB[N]$. Thus $A[N]B[N]$ is normal in G .

Finally this last lemma, which can be found in Schenkman [4], will turn out to be crucial.

LEMMA 4. *If $H = \langle a \rangle C$, where C is a normal Abelian subgroup and a is an element of H , then $H' = \{(a, c) | c \in C\}$.*

PROOF. Let $T = \{(a, c) | c \in C\}$. Then for any $c, c_0 \in C$, $(a, c)(a, c_0) = a^{-1}c^{-1}ac(a^{-1}c_0^{-1}a)c_0 = a^{-1}c^{-1}aa^{-1}c_0^{-1}acc_0 = (a, cc_0)$ since C is normal and Abelian. Also $(a, c)^{-1} = c^{-1}a^{-1}ca = a^{-1}cac^{-1} = (a, c^{-1})$ for the same reasons. Thus T is a subgroup.

Now H/C is Abelian, so we have $T \subseteq H' \subseteq C$. Thus to show that T is normal in H it suffices to show that T is invariant under conjugation by a . But $(a, c)^a = (a^a, c^a) = (a, c^a)$, so T is normal in H .

Clearly $\langle a \rangle T/T$ is in the center of H/T . But since C is Abelian, this means that H/T is Abelian and $H' \subseteq T$.

THEOREM. *If the group $G = AB$, where A and B are Abelian subgroups of G , $A \neq B$, and either A or B satisfies the maximum condition, then there is a normal subgroup N of G , $N \neq G$, such that $A \subseteq N$ or $B \subseteq N$.*

PROOF. Assume A satisfies the maximum condition.

Suppose the theorem is false. Let $G = AB$ be a counterexample.

(1) $G' \neq 1$, $A \neq G$, $B \neq G$, and $AG' = G = BG'$.

PROOF. $A \neq B$, and so either $G \neq A$ or $G \neq B$. If $G' = 1$, then G is Abelian and so one of the factors would be a proper normal subgroup of G . Thus $G' \neq 1$. Since A and B are Abelian, and G is not, we have $A \neq G$ and $B \neq G$. AG' and BG' are normal subgroups of G containing A and B respectively, so we must have $AG' = G = BG'$.

Now let $G = AB$ be a minimal counterexample in the following sense: If $G_0 = A_0B_0$ is any other counterexample, then $R(A) \leq R(A_0)$, and if $R(A) = R(A_0)$, then $T(A) \leq T(A_0)$. In addition:

(2) We may assume $B \cap G' = 1$.

PROOF. Let $D = B \cap G'$. Both B and G' are Abelian, so $G = BG' \subseteq C(D)$. Thus $D \subseteq Z$, and in particular, D is normal in G .

Suppose $AD = B$. Then we have $G = AB = AAD = AD = B$, and this is impossible by (1). So $AD \neq B$.

We now have $G/D = (AD/D)(B/D)$ and $AD/D \neq B/D$. Suppose there is a normal subgroup N/D of G/D , $N/D \neq G/D$, such that N/D contains either AD/D or B/D . Then N is normal in G , $N \neq G$, and N contains A or B . This

contradicts our assumption that the theorem was false for $G=AB$, and so $G/D=(AD/D)(B/D)$ must also be a counterexample.

Now $AD/D \cong A/A \cap D$. If $A \cap D \neq 1$, then either $R(AD/D) < R(A)$ or $R(AD/D)=R(A)$ and $T(AD/D) < T(A)$. In either case we have a smaller than minimal counterexample and this is impossible. So $A \cap D=1$, $AD/D \cong A$, and G/D is also a minimal counterexample.

In G/D we have $(G/D)'=G'/D$ since $D \subseteq G'$, and so $(G/D)' \cap (B/D)=1$. Hence, without loss of generality, we may assume $B \cap G'=1$.

(3) If N is a normal subgroup of G and $AN \neq BN$, then $A \cap N=1$.

PROOF. Suppose false. Then $A \cap N \neq 1$. But $AN/N=A/A \cap N$, and so either $R(AN/N) < R(A)$ or $R(AN/N)=R(A)$ and $T(AN/N) < T(A)$. In either case, $G/N=(AN/N)(BN/N)$ is a group satisfying the hypothesis of the theorem, which is smaller than a minimal counterexample. Thus there is a normal subgroup $U/N \neq G/N$ of G/N containing either AN/N or BN/N . But this means that U is a nontrivial normal subgroup of G , containing either A or B . Contradiction.

(4) $A \cap Z=1$.

PROOF. Suppose $AZ=BZ$. Then $G=AB=AAZ=AZ$. Both A and Z centralize A . Thus A is in the center of G and so $G=Z$. This contradicts $G' \neq 1$, so $AZ \neq BZ$. By (3), $A \cap Z=1$.

(5) $A \cap B=1$ and $A \cap G'=1$.

PROOF. A and B centralize $A \cap B$, so $A \cap B \subseteq A \cap Z=1$. A and G' centralize $A \cap G'$ since G' is Abelian by Lemma 2, and since $G=AG'$, $A \cap G' \subseteq A \cap Z=1$.

Now $A \neq 1$ since G is not Abelian. Let $a \neq 1$ be an element of A . Let $G_0 = \langle a \rangle G'$. $\langle a \rangle G'$ is normal in G and G_0 is characteristic in $\langle a \rangle G'$, so G_0 is normal in G . Clearly G_0 is a subgroup of G' , and so $A \cap G_0=1=B \cap G_0$.

(6) $G_0 \neq 1$.

PROOF. Suppose $G_0=1$. Then G' centralizes a . A is also in the centralizer of a , so $a \in A \cap Z$. But $a \neq 1$ and this contradicts (4).

Let $A_0=A[G_0]$ and $B_0=B[G_0]$. Let $\tilde{G}=A_0B_0=A_0G_0=B_0G_0$. By Lemma 3, $\tilde{G}$ is normal in G .

(7) $\tilde{G}=G$.

PROOF. Suppose false. $\tilde{G} \neq G$. Then

$$A\tilde{G} = AA_0B_0 = AB_0 \text{ and } B\tilde{G} = \tilde{G}B = A_0B_0B = A_0B.$$

Suppose $A\tilde{G}=B\tilde{G}$. Then $AB_0=A_0B$. By Lemma 1, $B=B_0(A \cap B)=B_0$. Thus B is a subgroup of a proper normal subgroup $\tilde{G}$, and this is impossible. Therefore $A\tilde{G} \neq B\tilde{G}$.

By (3), $A \cap \tilde{G}=1$. Thus $A_0=1$. $\tilde{G}=A_0B_0$, so $\tilde{G}=B_0$. But then $G_0 \subseteq B$, and thus $G_0=1$ since $G_0 \subseteq G'$. This contradicts (6) however, so (7) holds.

(8) $A_0=A$, $B_0=B$, and $G_0=G'$.

PROOF. $G = \bar{G}$, so $AB = A_0B_0$. By Lemma 1,

$$A = A_0(A \cap B_0) \subseteq A_0(A \cap B) = A_0.$$

Similarly $B = B_0$. We also have

$$G = AG' = A_0G_0, \text{ so } G' = G_0(A_0 \cap G') \subseteq G_0(A \cap G') = G_0,$$

so $G' = G_0$.

We can now conclude the proof. By Lemma 4, $G_0 = \{(a, g) | g \in G'\}$. But $a \in A = A_0$, so $a^{-1} \in A_0$. By definition, there is an element $b \in B$ such that $a^{-1}b \in G_0$. Thus there is an element $g \in G'$ such that $a^{-1}b = (a, g)$. So $b = g^{-1}ag$. $G = AB$ however so $g = a_0b_0$ for some $a_0 \in A$, $b_0 \in B$. Thus $b = b_0^{-1}a_0^{-1}aa_0b_0 = b_0^{-1}ab_0$. Conjugating by b_0^{-1} we get $1 \neq a = b \in A \cap B$. Contradiction.

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