A CORRECTION TO "ON FRICKE MODULI"

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ABSTRACT. In this paper, the "twist parameter" for combining Fuchsian groups is redefined correctly in terms of traces of the elements of the groups.

In [2], the Teichmüller space of a surface S of genus g is described real analytically in terms of traces of elements of the Fuchsian group G which represents S . An element of the Fuchsian group is determined by a free homotopy class of curves on the surface. The trace of the element is an analytic function of the length of the geodesic in this class. The group is constructed from the traces by first constructing subgroups corresponding to simple types of subsurfaces and then combining the subgroups as free products with amalgamation. These simple subsurfaces are conformally equivalent to either a torus from which a conformal disk has been removed (type $(1; 1)$) or to a sphere from which three conformal disks have been removed (type $(0; 3)$). In either case the subgroup is a free group on two generators. Combining these groups corresponds to attaching two of these surfaces across the boundaries of the removed disks or "holes". It is clear that the holes must have the same circumference, and hence that the corresponding group elements have the same trace. The conformal structure of the combined surface depends on one real parameter since the boundaries of the holes can be twisted against one another. The problem is to describe this parameter in terms of traces of elements of the resulting group, or equivalently, lengths of geodesics on the combined surfaces.

The full construction of a finitely generated Fuchsian group from traces of group elements is given in [2]. However, there is an ambiguity in the construction of this twist parameter. The purpose of this note is to clarify that ambiguity.

Let G_i , $i=1, 2$, be a Fuchsian group which represents a surface with a hole. Let C_i be an element of G_i which corresponds to the boundary curve of the hole. G_i is determined up to conjugation by a linear fractional transformation and hence it can be normalized. Let A_i be a generator of the simple subsurface to which C_i belongs. Normalize G_i so that the attracting

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fixed point of A_i is $+1$, the repelling fixed point of C_i is ∞ and the attracting fixed point of C_i is 0 . The interval $(-\infty, 0)$ is an interval of discontinuity for G_i .

Let $R(x)$ be a linear fractional transformation which sends $(0, \infty, 1)$ to $(\infty, 0, x)$, $x < 0$. Let $G_2(x) = R(x)G_2R^{-1}(x)$. $(0, \infty)$ is an interval of discontinuity for $G_2(x)$ and $\{G(x) = G_1 * G_2(x) \text{ am } H, x < 0\}$, where $H = \langle C_2 = C_1^{-1} \rangle$ is the one parameter family of combined Fuchsian groups.

This parametrization is not the desired one since x is not the trace of a group element. In [2], it is erroneously stated that prescribing $\tau = \text{trace } A_2(x)A_1$ uniquely determines x . In fact, prescribing τ determines only the sine of the angle of intersection of the axes of C_1 and $A_2(x)A_1$. This is not enough since it is necessary to determine this angle exactly.

This difficulty can be circumvented as follows. In the above normalization, A_1 , $A_2(x)$ and C_1 can be expressed as:

$$A_1 = \begin{pmatrix} r_1 + R_1\xi_1 & R_1(1 - \xi_1) \\ R_1(1 + \xi_1) & r_1 - R_1\xi_1 \end{pmatrix},$$

$$A_2(x) = \begin{pmatrix} r_2 - R_2\xi_2 & xR_2(1 + \xi_2) \\ \frac{R_2}{x}(1 - \xi_2) & r_2 + R_2\xi_2 \end{pmatrix},$$

$R_i = (r_i^2 - 1)^{1/2}$, $r_i, \xi_i > 1$. r_i and ξ_i are determined in the construction of G_i .

$$C_1 = C_2^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad |\lambda| < 1.$$

x determines:

$$(1) \quad \text{tr } A_2(x)A_1 = \tau = 2r_1r_2 + 2R_1R_2\xi_1\xi_2 + (R_1R_2/x)(1 - \xi_1)(1 - \xi_2) + R_1R_2x(1 + \xi_1)(1 + \xi_2)$$

and

$$(2) \quad \text{tr } C_1A_2(x)A_1 = \sigma = (\lambda + 1/\lambda)(r_1r_2 + R_1R_2\xi_1\xi_2) + (\lambda - 1/\lambda)(r_2R_1\xi_1 - r_1R_2\xi_2) + \frac{\lambda R_1R_2(1 - \xi_1)(1 - \xi_2)}{x} + \frac{R_1R_2x(1 + \xi_1)(1 + \xi_2)}{\lambda}$$

Let $\tilde{\tau} = \tau - 2(r_1r_2 + R_1R_2\xi_1\xi_2)$ and

$$\tilde{\sigma} = \sigma - (\lambda + 1/\lambda)(r_1r_2 + R_1R_2\xi_1\xi_2) - (\lambda - 1/\lambda)(r_2R_1\xi_1 - r_1R_2\xi_2)$$

then $\tilde{\sigma}$ and $\tilde{\tau}$ satisfy

$$(*) \quad \tilde{\sigma}^2 - (\lambda + 1/\lambda)\tilde{\sigma}\tilde{\tau} + \tilde{\tau}^2 = -R_1^2R_2^2(1 - \xi_1^2)(1 - \xi_2^2).$$

Given σ and τ (hence $\bar{\sigma}$ and $\bar{\tau}$) satisfying the relation (*) there is a unique x such that (1) and (2) hold. This then is a solution to the problem of describing the combination parameter in terms of traces of elements of the Fuchsian group.

REMARK (1). For a given pair of groups G_1 and G_2 , τ will vary in absolute value from some number T , which depends on G_1 and G_2 , to ∞ . T is determined from the turning point on the hyperbola (*). $|T|$ is bounded away from 2 unless there is some degeneration in either of the groups G_1 or G_2 . This will be discussed more fully in [3].

REMARK (2). To each τ correspond two σ 's. In general the resulting surfaces are not conformally equivalent. They are conjugate to one another through a reflection in the Teichmüller space across the curve $\tau = T$.

BIBLIOGRAPHY

1. R. Fricke and F. Klein, *Vorlesungen über die Theorie der Automorphen Funktionen*, Leipzig, 1926.
2. L. Keen, *On Fricke moduli*, Advances in the Theory of Riemann Surfaces, Princeton Univ. Press, Princeton, N.J., 1971.
3. L. Keen, *A fundamental domain for the modular group for Riemann surfaces of genus 2* (to appear).

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