A NOTE ON MULTIVALENCE OF FUNCTIONS OF BOUNDED INDEX

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ABSTRACT. This paper examines the relationship between the concept of bounded index and the radius of univalence, respectively p -valence, of entire functions and their derivatives at arbitrary points in the plane.

1. Introduction. An entire function $f(z)$ is of bounded index if there exists an integer N , independent of z , such that

$$\max_{0 \leq l \leq N} \left\{ \frac{|f^{(l)}(z)|}{l!} \right\} \geq \frac{|f^{(n)}(z)|}{n!} \quad \text{for all } n.$$

The least such integer N is called the index of $f(z)$. The concept of functions of bounded index was first introduced by Lepson [4].

For an entire function f and a complex number w let $r(w, f)$ denote the radius of univalence of $f(z+w)$. Let $R_N(w) = R_N(w, f) = \max_{0 \leq i \leq N} \{r(w, f^{(i)})\}$. Then we prove the following results.

THEOREM 1. *Let f be an entire function. Then f' is of bounded index if and only if there exist an integer $N > 0$ and a constant $\delta > 0$ such that $R_N(w) \geq \delta$ for all $w \in \mathbb{C}$, i.e. for any point in the plane, f or one of the first N derivatives has radius of univalence of at least δ .*

THEOREM 2. *Let f be an entire function. Then f' is of bounded index if and only if there exists an integer $p > 0$ such that f is p -valent in any disc of radius 1.*

2. Proof of Theorem 1. " $\Rightarrow$ " Let f' be of bounded index. Then, by [2, Theorem 2], $f'(\alpha z)$ is of bounded index for any $\alpha \in \mathbb{C}$. Let N be the index of $f'(2z)$ and, for $w \in \mathbb{C}$, let $f(z) = \sum_{n=0}^{\infty} a_n(z-w)^n$.

Hence, there exists an integer k , $0 \leq k \leq N$, such that

$$|a_{k+1}| \geq (k+1)^{-1}(k+1+j)2^j |a_{k+1+j}| \geq 2^j |a_{k+1+j}| \quad \text{for } j = 1, 2, \dots$$

Presented to the Society, November 20, 1972; received by the editors November 28, 1972.

AMS (MOS) subject classifications (1970). Primary 30A66; Secondary 30A64.

Key words and phrases. Functions of bounded index, univalent, p -valent, exponential type.

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Now,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (z-w)^{n-k} = \sum_{j=0}^{\infty} b_j (z-w)^j.$$

Therefore, for $j=1, 2, \dots$,

$$\begin{aligned} \frac{|b_{j+1}|}{|b_1|} &= \frac{(k+1+j)(k+j) \cdots (j+2)}{(k+1)!} \frac{|a_{k+1+j}|}{|a_{k+1}|} \\ &\leq (j+1)^k \frac{|a_{k+1+j}|}{|a_{k+1}|} \leq \frac{(j+1)^k}{2^j}. \end{aligned}$$

Thus,

$$\sum_{n=2}^{\infty} n |b_n| \leq \sum_{j=1}^{\infty} \frac{(j+1)^{k+1}}{2^j} |b_1| \leq |b_1| \sum_{j=1}^{\infty} (j+1) N^{+1} 2^{-j} = |b_1| B,$$

where $B > 0$ is a constant independent of w . It is well known that $\sum_{n=2}^{\infty} n |c_n| < 1$ implies $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ is starlike univalent in $|z| < 1$, (compare [1]). Therefore, $f^{(k)}(z) = \sum_{j=0}^{\infty} b_j (z-w)^j$ is univalent in $|z-w| < 1/B$.

“ $\Leftarrow$ ” Choose $N > 0$ and $\delta > 0$ such that $R_N(w) \geq \delta$ for all $w \in C$. Then, given $w \in C$, there exists an integer $k \leq N$ such that $f^{(k)}(z) = \sum_{j=0}^{\infty} b_j (z-w)^j$ is univalent in $|z-w| < \delta$. Obviously $b_1 \neq 0$ and therefore

$$g(z) = \sum_{n=1}^{\infty} \frac{b_n}{b_1} \delta^{n-1} z^n = \sum_{n=1}^{\infty} c_n z^n$$

is univalent in $|z| < 1$ with $c_1 = 1$. It is well known that then $|c_n| < en$ for $n=2, 3, \dots$, (see [3, Theorem 1.6]). Hence, $|b_1| \geq (|b_n|/en) \delta^{n-1}$ for $n=1, 2, \dots$. Now, if $f(z) = \sum_{t=0}^{\infty} a_t (z-w)^t$ then

$$|a_{k+1}| = \frac{|b_1|}{(k+1)!} \geq \frac{|b_n|}{(k+1)!} \frac{\delta^{n-1}}{en} \geq |a_{k+n}| \frac{\delta^{n-1}}{en} \quad \text{for } n = 1, 2, \dots.$$

Hence, $f'(z/T)$ is of index not exceeding N for T sufficiently large and thus $f'(z)$ is of bounded index. Q.E.D.

3. Proof of Theorem 2. “ $\Rightarrow$ ” Let f' be of index N , let b be a root of $f(z) = w$, and let $f(z) = w + \sum_{n=1}^{\infty} a_n (z-b)^n$. Since f' is of index N , there exists an integer k with $1 \leq k \leq N+1$ such that $|a_k| \geq |a_j|$ for $j=1, 2, \dots$. Let $P(z) = \sum_{n=1}^k (a_n/a_k) z^n$. Then $P(z) = \prod_{j=1}^k (z-c_j)$ for some $c_j \in C$. Let $T = \{4(N+1)\}^{N+1}$. Then there exists a constant r with $(2kT)^{-1} \leq r \leq 1/T$ such that for z with $|z| = r$ we have $|z-c_j| \geq (2kT)^{-1}$ for $j=1, \dots, k$. Thus $|P(z)| \geq (1/2kT)^k$ for all $|z| = r$. Now, for $|z| = r$,

$$\left| \sum_{n=k+1}^{\infty} \frac{a_n}{a_k} z^n \right| \leq \sum_{n=k+1}^{\infty} \left| \frac{a_n}{a_k} z^n \right| \leq \sum_{n=k+1}^{\infty} |z|^n \leq \sum_{n=k+1}^{\infty} T^{-n} < \frac{2}{T^{k+1}}.$$

Therefore,

$$|P(z)| \geq (2kT)^{-k} \geq \frac{2}{T^{k+1}} > \left| \sum_{n=k+1}^{\infty} \frac{a_n}{a_k} z^n \right| \quad \text{for all } |z| = r.$$

By Rouché's theorem, $\sum_{n=1}^{\infty} (a_n/a_k)z^n$ has the same number of zeros in $|z| < r$ as $P(z)$. Hence $\sum_{n=1}^{\infty} a_n z^n$ has at most k zeros in $|z| < (2kT)^{-1}$ and $f(z) = w$ has at most k solutions in $|z - b| < (2kT)^{-1}$. In general, $f(z)$ is at most $(N+1)$ -valent in $|z - \gamma| < \{4(N+1)T\}^{-1}$ for all $\gamma \in C$. Thus there exists $p \geq N+1$ such that f is p -valent in any disc of radius 1.

" $\Leftarrow$ " Let $f(z) = \sum_{n=0}^{\infty} a_n(z-w)^n$ be p -valent in $|z-w| < 1$. Without loss of generality we may assume $a_0 = 0$. Then, by [2, Theorem 3.5] we have, for $n = 1, 2, \dots$, $|a_n| < A(p) \max_{1 \leq v \leq p} \{|a_v|\} n^{2p-1}$, where $A(p)$ depends only on p . Thus, in general,

$$\frac{|f^{(n)}(z)|}{n!} < A(p) n^{2p-1} \max_{1 \leq v \leq p} \left\{ \frac{|f^{(v)}(z)|}{v!} \right\} \quad \text{for } n = 1, 2, \dots$$

Hence, $f'(z/T)$ is of index p for T sufficiently large and therefore $f'(z)$ is of bounded index. Q.E.D.

REFERENCES

1. J. Clunie and F. R. Keogh, *On starlike and convex schlicht functions*, J. London Math. Soc. **35** (1960), 229–233. MR **22** #1682.
2. G. H. Fricke, *A characterization of functions of bounded index*, Indian J. Math. (to appear).
3. W. K. Hayman, *Multivalent functions*, Cambridge Tracts in Math. and Math. Phys., no. 48, Cambridge Univ. Press, Cambridge, 1958. MR **21** #7302.
4. B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, Entire Functions and Related Parts of Analysis, Proc. Sympos. Pure Math. 11, Amer. Math. Soc., Providence, R.I., 1968, pp. 298–307. MR **38** #6069.

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