THE SOLUTION OF AN INTEGRAL EQUATION

C. NASIM

ABSTRACT. Various methods are developed to solve the integral equation $f(x) = \int_0^\infty g(t)k(xt) dt$, when the Mellin transform $K(s)$ of the kernel function $k(x)$ is decomposable. Each method corresponds to the way $K(s)$ is decomposed: Namely (i) $K(s) = 1/L(1-s)M(1-s)$, (ii) $K(s) = H(s)/M(1-s)$ and (iii) $K(s) = N(s)H(s)$, where L , M , N and H are arbitrary functions of the complex variable s . Numerous special cases and examples are given to illustrate the technique and the advantage of these methods.

1. Introduction. Let us consider the integral equation

$$(1) \quad g(x) = \int_0^\infty f(t)k(xt) dt, \quad x > 0,$$

where the function $g(x)$ and $k(x)$ are known and $f(x)$ is to be found. Many techniques are known to solve this integral equation, depending upon the properties which the Mellin transform of the function $k(x)$, called the kernel, must satisfy. For example, the most familiar case is the classical one, where $k(x)$ is the Fourier kernel [2, Chapter III], that is, its Mellin transform $K(s)$, satisfies the following functional equation

$$K(s)K(1-s) = 1,$$

where $s = c + it$, $-\infty < t < \infty$, for some real c , and $k(x)$ in some sense is given by

$$k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s)\bar{x}^s ds.$$

Then, by the Fourier inversion formula, the solution is

$$(x) = \int_0^\infty k(xt)g(t) dt.$$

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If $K(s)$ is a rational function involving gamma function, i.e.

$$K(s) = \prod_{i=1}^m \Gamma(\alpha_i s + \beta_i) / \prod_{j=1}^n \Gamma(\alpha_j s + \beta_j),$$

then by applying the operators L and L^{-1} , to both sides of (1), L being the Laplace operator, the solution can be found [1]. Also, the integral equation (1) is solvable in terms of Mellin transforms and the solution is,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(1-s)}{K(1-s)} \bar{x}^s ds,$$

where $G(s)$ is the Mellin transform of $g(x)$.

In this paper we shall give a method for solving the integral equation (1), when $K(s)$ is decomposable into arbitrary functions of s . The cases which we shall consider here are when $K(s)$ is decomposable into the following forms:

(i) $K(s) = 1/L(1-s)M(1-s),$

(ii) $K(s) = H(s)/M(1-s),$

(iii) $K(s) = N(s)H(s),$

where L, M, N and H are arbitrary functions of s ; $s = c + it$, $-\infty < t < \infty$.

2. Preliminaries. For simplicity, we shall confine ourselves to L^2 -space and base the proofs given here upon the Mellin transforms. The following well-known theorems will be used and are found in [2, pp. 94, 95].

A. If $f(x) \in L^2(0, \infty)$, then

$$F(s) = \int_0^\infty f(t)t^{s-1} dt,$$

the integral converging to $F(s)$ in the mean square sense and $F(s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$.

If $F(s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, then

$$f(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)\bar{x}^s ds,$$

the integral converging to $f(x)$ in the mean square sense and $f(x) \in L^2(0, \infty)$.

This defines $F(s)$ to be the Mellin transform of $f(x)$.

B. THE PARSEVAL THEOREM. If $f(x)$ and $g(x)$ both belong to $L^2(0, \infty)$, and $F(s), G(s)$ are their respective Mellin transforms, then

$$\int_0^\infty f(x)g(x) dx = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)G(1-s) ds.$$

C. If $f(x)$ and $g(x)$ both belong to $L^2(0, \infty)$, then by the results (A) and (B), the integral equation (1) gives

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} G(s) \bar{x}^s ds = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s) K(s) \bar{x}^s ds.$$

Therefore

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} [G(s) - F(1-s)K(s)] \bar{x}^s ds = 0,$$

and

$$(2) \quad G(s) = F(1-s)K(s)$$

a.e. on $s = \frac{1}{2} + it$, $-\infty < t < \infty$.

3. Theorems.

THEOREM 1. If $K(s) = 1/L(1-s)M(1-s)$, $s = \frac{1}{2} + it$, $-\infty < t < \infty$, then define $\Phi(s)$ as $\Phi(s) = L(s)G(1-s)$. Then the solution of the integral equation (1) is

$$f(x) = \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt, \quad x > 0,$$

where $K(s)$, $\Phi(s)$ and $M(s)$ are the respective Mellin transforms of the functions $k(x)$, $\varphi(x)$ and $m(x)$, according to the result (A).

PROOF. Since $\varphi(x) \in L^2(0, \infty)$, by hypothesis, therefore it is an easy matter to see that $(1/x)\varphi(1/x) \in L^2(0, \infty)$ also.

Now $m(x)$ and $(1/x)\varphi(1/x)$ both belong to $L^2(0, \infty)$, therefore the integral $\int_0^\infty (1/t)\varphi(1/t)m(xt) dt$, is absolutely convergent for $x > 0$. By applying the Parseval Theorem (B) for Mellin transforms of L^2 -functions, we obtain

$$\int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s) M(s) \bar{x}^s ds.$$

Now due to the result (C) we have

$$G(s) = F(1-s)K(s) = F(1-s)/L(1-s)M(1-s).$$

Therefore $F(s) = G(1-s)L(s)M(s) = \Phi(s)M(s)$. And

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s) M(s) \bar{x}^s ds = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s) \bar{x}^s ds = f(x),$$

in the mean square sense by the result (A), as required.

THEOREM 2. If $K(s) = H(s)/M(1-s)$, $s = \frac{1}{2} + it$, $-\infty < t < \infty$, then define $\Phi(s)$ to be $\Phi(s) = G(1-s)/H(1-s)$. Then the solution of the integral equation

(1) is

$$f(x) = \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt, \quad x > 0,$$

where $K(s)$, $\Phi(s)$ and $M(s)$ are as in Theorem 1.

PROOF. Again $(1/x)\varphi(1/x)$ and $m(x)$ both belong to $L^2(0, \infty)$, therefore by the result (B),

$$\int_0^\infty \frac{1}{t} \Phi\left(\frac{1}{t}\right) m(xt) dt = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s) M(s) \bar{x}^s ds.$$

Also, due to (C),

$$G(s) = F(1-s)K(s) = F(1-s)H(s)/M(1-s),$$

and therefore

$$F(s) = G(1-s)M(s)/H(1-s) = \Phi(s)M(s).$$

Thus

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s) M(s) \bar{x}^s ds = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s) \bar{x}^s ds = f(x)$$

in the mean square, by the result (A), as required.

In the case when $K(s) = N(s)H(s)$, if we define $\Phi(s) = G(1-s)/H(1-s)$, then it is easy to see that the solution of the integral equation $\Phi(x) = (1/x) \int_0^\infty f(u) n(u/x) du$ is also the solution of (1).

Note 1. In Theorem 1, if $L(s)M(s) = H(s)$, then we have the functional relation $K(s)H(1-s) = 1$. The solution of (1) is then given by

$$f(x) = \int_0^\infty g(t) h(xt) dt,$$

which is the usual unsymmetrical Fourier integral formula [2, Chapter VIII]. Numerous examples of the relation between $h(x)$ and $k(x)$ can be found in the literature [2, Chapter VIII].

Note 2. In Theorem 1, if $L(s)L(1-s) = 1$, then $K(s) = L(s)/M(1-s)$ which is the case considered in Theorem 2.

Note 3. In Theorem 2, if $H(s)H(1-s) = M(s)M(1-s)$, then

$$K(s)K(1-s) = 1,$$

and then $k(x)$ is the Fourier kernel for the pair of functions $f(x)$ and $g(x)$. Hence the solution of (1) is

$$f(x) = \int_0^\infty g(t) k(xt) dt.$$

The results mentioned above will also hold in the function space $L(0, \infty)$. For this we need to impose appropriate conditions on the functions involved so that the results corresponding to (A) and (B) can be applied. These can be found in [2, pp. 46, 60].

4. Examples. The methods given above to solve the integral equation (1), essentially depend upon the decomposition of $K(s)$. This is done in such a way, so that the following integrals can be evaluated to give, respectively,

$$(i) \quad (1/2\pi i) \int_{1/2-i\infty}^{1/2+i\infty} M(s)\bar{x}^s ds = m(x),$$

$$(ii) \quad (1/2\pi i) \int_{1/2} \Phi(s)\bar{x}^s ds = \varphi(x), \text{ and}$$

$$(iii) \quad \int_0^\infty (1/t)\varphi(1/t)m(xt) dt = f(x).$$

The advantage of these methods lies, therefore, in assuming that these above integrals involve elementary functions and are easy to evaluate or can simply be read from the tables ([3], [4]). To confirm this assumption, we give below a few examples. Throughout these examples the Mellin transforms are defined for the appropriate range of $\text{Re}(s)$.

1. *Laplace's integral equation.*

$$g(x) = \int_0^\infty f(t)\bar{e}^{xt} dt.$$

(i) Here $k(x) = \bar{e}^x$ and $K(s) = \Gamma(s) = \pi/\Gamma(1-s)\sin \pi s$, where $L(s) = \sin \pi s$ and $M(s) = \Gamma(s)$. Thus $m(x) = \bar{e}^x$.

If $g(x) = \log x/(x-1)$, so that $G(s) = \pi^2 \text{cosec}^2 \pi s$ [4, p. 538], then

$$\Phi(s) = L(s)G(1-s) = \pi \text{cosec } \pi s,$$

whence $\varphi(x) = 1/(1+x)$ [3, p. 308]. Thus by Theorem 1,

$$f(x) = \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt = \int_0^\infty \frac{\bar{e}^{xt}}{1+t} dt = \Gamma(0, x)e^x \quad [3, p. 137].$$

2. *Stieltjes's integral equation.*

$$\Psi(x) = \int_0^\infty \frac{f(u)}{x+u} du.$$

By change of variable, we can write this as

$$g(x) = \frac{1}{x} \Psi\left(\frac{1}{x}\right) = \int_0^\infty \frac{f(x)}{1+xt} dt.$$

(i) Consider $k(x) = 1/(1+x)$. Then

$$K(s) = \pi \text{cosec } \pi s = \frac{\pi \Gamma(1-s)}{2\Gamma(1-s)\sin \frac{1}{2}\pi s \cos \frac{1}{2}\pi s}$$

where $H(s) = \Gamma(1-s) \operatorname{cosec} \frac{1}{2}\pi(1-s)$, and $M(s) = -(2/\pi)\Gamma(s) \cos \frac{1}{2}\pi s$, thus $m(x) = -(2/\pi) \cos x$ [3, p. 348].

Now if $\Psi(x) = K_0(2x^{1/2})$, then $g(x) = \bar{x}^1 K_0(2x^{-1/2})$, and $G(s) = \Gamma^2(1-s)$, and $\Phi(s) = G(1-s)/H(1-s) = \Gamma(s) \sin \frac{1}{2}\pi s$, whence $\varphi(x) = \sin x$.

Thus by Theorem 2

$$f(x) = \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt = \frac{-2}{\pi} \int_0^\infty \frac{1}{t} \sin\left(\frac{1}{t}\right) \cos xt dt = -J_0(2x^{1/2})$$

[3, p. 25].

(ii) Now

$$K(s) = \pi \operatorname{cosec} \pi s = \frac{\pi \operatorname{cosec} \pi s \Gamma(1-s)}{\Gamma(1-s)},$$

where $H(s) = \pi \Gamma(1-s) \operatorname{cosec} \pi s$ and $M(s) = \Gamma(s)$. Thus $m(x) = \bar{e}^x$.

If $\Psi(s) = \log x/(x-1)$, then $g(x) = \log x/(x-1)$, and $G(s) = \pi^2 \operatorname{cosec}^2 \pi s$ [4, p. 538].

We define, $\Phi(s) = G(1-s)/H(1-s) = \Gamma(1-s)$, whence $\Phi(x) = \bar{x}^1 \bar{e}^{1/x}$. Thus by Theorem 2,

$$f(x) = \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt = \int_0^\infty \bar{e}^{t(1+x)} dt = \frac{1}{1+x}.$$

3. Cosine integral equation.

$$g(x) = \int_0^\infty f(t) \cos xt dt.$$

Here $k(x) = \cos x$, therefore $K(s) = \Gamma(s) \cos \frac{1}{2}\pi s = \Gamma(s) / \sec \frac{1}{2}\pi s$, where $H(s) = \Gamma(s)$ and $M(s) = \operatorname{cosec} \frac{1}{2}\pi s$. Thus $m(x) = 1/(1+x^2)$ [3, p. 345].

Now, if $g(x) = 1/(1+x)$, then $G(s) = \operatorname{cosec} \pi s$, and

$$\Phi(s) = G(1-s)/H(1-s) = \Gamma(s),$$

whence $\varphi(x) = \bar{e}^x$. Thus by Theorem 2,

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt = \int_0^\infty \frac{u}{u^2 + x^2} \bar{e}^u du \\ &= ci(x) \cos x - si(x) \sin x \quad [4, p. 312]. \end{aligned}$$

4. Let $k(x) = J_v(u)$, and therefore

$$K(s) = 2^{s-1} \frac{\Gamma(\frac{1}{2}s + \frac{1}{2}v)}{\Gamma(\frac{1}{2}v - \frac{1}{2}s + 1)}, \quad -R(v) < R(s) < \frac{3}{2},$$

where

$$H(s) = 2^{s-1} \Gamma(\frac{1}{2}s + \frac{1}{2}v) \quad \text{and} \quad M(s) = \Gamma(\frac{1}{2}v + \frac{1}{2}s + \frac{1}{2}),$$

whence $m(x) = 2x^{v+1}\tilde{e}^{x^2}$. Now if $g(x) = K_v(x)$, then $\Phi(s) = G(1-s)/H(1-s)$ gives $\varphi(x) = x^{v-1}e^{-1/x^2}$. Thus

$$f(x) = \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt = 2x^{v+1} \int_0^\infty t \tilde{e}^{t^2(1+x^2)} dt = \frac{x^{v+1}}{1+x^2},$$

$$-1 < R(v) < \frac{3}{2}.$$

More generally if $g(x) = x^\mu K_{v-\mu}(x)$, then $f(x) = 2^\mu \Gamma(\mu+1) x^{v+1}/(1+x^2)^{\mu+1}$
 5. Let $k(x) = x^{1/2} H_v(x)$; then [3, p. 335] $K(s) = H(s)/M(1-s)$ where

$$H(s) = \tan \frac{\pi}{2} (s + v + \tfrac{1}{2}) \quad \text{and} \quad M(s) = 2^{s-1/2} \frac{\Gamma(\frac{1}{2}v + \frac{1}{2}s + \frac{1}{4})}{\Gamma(\frac{1}{2}v - \frac{1}{2}s + \frac{3}{4})}$$

whence $m(x) = x^{1/2} J_v(x)$.

Now if $g(x) = x^{v-1/2}/(1+x^2)$, then $\Phi(s) = G(1-s)/H(1-s)$ gives $\varphi(x) = x^{1/2-v}/(1+x^2)$. Thus by Theorem 1,

$$f(x) = \int_0^\infty \frac{1}{t} \varphi\left(\frac{1}{t}\right) m(xt) dt = x^{1/2} \int_0^\infty \frac{t^{v+1}}{1+t^2} J_v(xt) dt = x^{1/2} K_v(x),$$

$$-1 < R(v) < \frac{3}{2} \quad [4, \text{p. 686}].$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA