A CHARACTERISTIC EXTREMAL PROPERTY OF SIMPLICES

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ABSTRACT. Among all d -dimensional convex bodies the simplices are characterized by the property that the volume of every circumscribed cylinder is not less than d times the volume of the body itself.

For every convex body K with interior points in d -dimensional Euclidean space E^d there exists a cylinder Z with the properties $K \subseteq Z$ and $V(Z) \leq dV(K)$ (as follows from Lemma 6 of Macbeath [6]). Here V denotes the volume, and by a cylinder in E^d we understand any d -dimensional convex body which is the Minkowski sum of a $(d-1)$ -dimensional convex body and a line segment. Butler [3] has shown that $V(Z) \geq dV(K)$ for every simplex K in E^d and all cylinders Z circumscribed about K . Hence the functional f (a "measure of noncylindricity", as it might be called) defined by

$$f(K) = \inf\{V(Z)/V(K) : K \subseteq Z \text{ and } Z \text{ a cylinder}\}$$

for every d -dimensional convex body K , attains its maximum if K is a simplex. Butler [3, Remark 3, p. 560] left open the question whether simplices are the only extremal bodies. This question is answered affirmatively by the following result.

Suppose $K \subset E^d$ is a d -dimensional convex body which satisfies $V(Z) \geq dV(K)$ for every circumscribed cylinder Z . Then K is a simplex.

PROOF. We denote the convex hull of a set M by $\text{conv } M$ and its affine hull by $\text{aff } M$. Suppose K is a d -dimensional convex body in E^d . Choose a regular boundary point p of K . Through p there exists a unique supporting hyperplane H of K . Let H^* be the other supporting hyperplane of K parallel to H . Choose a point $q \in K \cap H^*$ and let C be the infinite convex cylinder consisting of all lines parallel to $L_0 = \text{aff}\{p, q\}$ which meet the body K . Write $B = C \cap H$ and $P = \text{conv}(B \cup \{q\})$. If L_1 is a line parallel to L_0 and different from L_0 which meets the interior of K , then the set of real numbers λ for which the line $L_\lambda = (1-\lambda)L_0 + \lambda L_1$ meets K contains a

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greatest element Λ . If b is the intersection point of L_Λ and H , then

$$(1) \quad L_1 \cap P = L_1 \cap \text{conv}\{p, q, b\}.$$

If this were false, there existed a point $x \in L_1 \cap P$ such that $x \notin L_1 \cap \text{conv}\{p, q, b\}$. The point x is contained in the convex hull of q and some point $y \in B$. The line through y parallel to L_0 meets K . But this contradicts the maximality of L_Λ , since clearly the segment $\text{conv}\{p, y\}$ contains the point b in its relative interior.

Choose a point $a \in L_\Lambda \cap K$. As

$$(2) \quad L_1 \cap K \supseteq L_1 \cap \text{conv}\{p, q, a\}$$

and the segments on the right-hand sides of (1) and (2) are equal in length, it follows that $L_1 \cap P$ is not longer than $L_1 \cap K$. Since L_1 may be chosen arbitrarily in C we deduce

$$(3) \quad V(P) \leq V(K).$$

Let Z denote the compact cylinder which is cut out from C by the hyperplanes H and H^* . From $V(Z) = dV(P)$ we deduce $V(Z) \leq dV(K)$.

Now suppose that K satisfies $V(Z) \geq dV(K)$ for every circumscribed cylinder Z . Then equality must hold in (3). This implies

$$(4) \quad L_1 \cap K = L_1 \cap \text{conv}\{p, q, a\},$$

hence the segment $\text{conv}\{p, a\}$ contains a boundary point c of K in its relative interior. Then each supporting hyperplane of K through c contains p and therefore coincides with H . This shows $a \in H$, hence $a=b$. From (1), (4) and the arbitrariness of L_1 we deduce $K=P$. Since p was an arbitrary regular boundary point of K , we have shown that K has the following property:

(5) *For every regular boundary point p of K there exist a supporting hyperplane H of K and a point q of K such that $p \in H$ and*

$$K = \text{conv}((H \cap K) \cup \{q\}).$$

Let H and q be as in (5). As K has interior points, the face $H \cap K$ is $(d-1)$ -dimensional. Choose a point p' which is a regular boundary point of $H \cap K$ if this set is considered as a full-dimensional convex set in H . Then every point r in the relative interior of the segment $\text{conv}\{p', q\}$ is a regular boundary point of K . By (5) there exist a supporting hyperplane J of K and a point q' of K with the properties $r \in J$ (which implies $p' \in J$) and $K = \text{conv}((J \cap K) \cup \{q'\})$. Since q' is contained in the convex hull of q and some point s of H , and s is contained in the convex hull of q' and

some point of J , we have $q' = s \in H$. We deduce

$$H \cap K = H \cap \text{conv}((J \cap K) \cup \{q'\}) = \text{conv}((H' \cap H \cap K) \cup \{q'\}),$$

where $H' = J \cap H$ is a $(d-2)$ -dimensional flat in H which supports $H \cap K$ and contains p' . We have proved that $H \cap K$, considered as a body in $(d-1)$ -dimensional space, has property (5). Now we can use induction with respect to the dimension to show that a d -dimensional convex body satisfying (5) is a simplex; the case $d=2$ is, in fact, trivial.

REMARKS. For $d=2$ our result is contained implicitly in investigations of Estermann [4, p. 472] and Levi [5, p. 126].

It should be mentioned that our method of proof is related to the process of "shakedown" ("Schüttelung"), which has proved useful in establishing other extremal properties of simplices; see, e.g., Blaschke [1, p. 58], Bonnesen-Fenchel [2, p. 71].

REFERENCES

1. W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. II, Springer, Berlin, 1923.
2. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934.
3. G. J. Butler, *An extremal property of simplices*, Proc. Amer. Math. Soc. **30** (1971), 556–560. MR **44** #2140.
4. T. Estermann, *Über den Vektorenbereich eines konvexen Körpers*, Math. Z. **28** (1928), 471–475.
5. F. W. Levi, *Über zwei Sätze von Herrn Besicovitch*, Arch. Math. **3** (1952), 125–129. MR **14**, 309.
6. A. M. Macbeath, *A compactness theorem for affine equivalence-classes of convex regions*, Canad. J. Math. **3** (1951), 54–61. MR **13**, 577.

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