THE PROJECTIVE CLASS GROUP OF THE FUNDAMENTAL GROUP OF A SURFACE IS TRIVIAL¹

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ABSTRACT. Let $D = F_1 \times F_2 \times \cdots \times F_n$ be a direct product of n free groups $F_1, F_2, \dots, F_n$, α an automorphism of D which leaves all but one of the noncyclic factors in D pointwise fixed and T an infinite cyclic group. Let $D \times_{\alpha} T$ be the semidirect product of D and T with respect to α . We prove that the Whitehead group of $D \times_{\alpha} T$ and the projective class group of the integral group ring $Z(D \times_{\alpha} T)$ are trivial. The second result implies that the projective class group of the integral group ring over the fundamental group of a surface is trivial.

Let G be a group. We denote the Whitehead group of G by $\text{Wh } G$ and the projective class group of the integral group ring $Z(G)$ of G by $\tilde{K}_0 Z(G)$. Let α be an automorphism of G and let T be an infinite cyclic group. Then we denote by $G \times_{\alpha} T$ the semidirect product of G and T with respect to α .

Let M be a connected 2-dimensional manifold and $\pi_1(M)$ the fundamental group of M . If M is open, then $\pi_1(M)$ is a free group so that $\tilde{K}_0 Z(\pi_1(M))$ is trivial by a theorem of Bass (cf. [1]). Next, if M is a sphere or a projective plane, then $\pi_1(M) = 0$ or T_2 (cyclic group of order 2) and so $\tilde{K}_0 Z(\pi_1(M)) = 0$ (cf. [7, p. 419]). Now, if M is closed and is not a sphere or projective plane, then Farrell-Hsiang [4] have shown that $\pi_1(M)$ is just the semidirect product $F \times_{\alpha} T$, where F is a free group. The purpose of this paper is to show that $\tilde{K}_0 Z(F \times_{\alpha} T) = 0$ and so the projective class group of the integral group ring over the fundamental group of a surface is always trivial. In fact, we prove:

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THEOREM A. *Let $D = F_1 \times F_2 \times \cdots \times F_n$ be a direct product of n free groups $F_1, F_2, \dots, F_n$ and α an automorphism of D which leaves all but one of the noncyclic factors in D pointwise fixed. Then $\tilde{K}_0 Z(D \rtimes_{\alpha} T) = 0$.*

When $n=1$, this implies the result mentioned above.

In [4], Farrell-Hsiang also proved that $\text{Wh}(F \rtimes_{\alpha} T) = 0$. We will generalize it to:

THEOREM B. *Let D and α be as given in Theorem A. Then $\text{Wh}(D \rtimes_{\alpha} T) = 0$.*

To prove Theorem B, we need the following:

LEMMA 1. *Let D be as given in Theorem A. Then $\text{Wh } D = 0$ and $\tilde{K}_0 Z(D) = 0$.*

Before proving Lemma 1, we recall some definitions and results from [4] and [3]. For more details, we refer to [4] and [3].

Let R be an associative ring with identity and α an automorphism of R . We denote by $R_{\alpha}[T]$, the α -twisted finite Laurent series ring of R . Recall that an additive map ϕ from a right R -module M_1 to a right R -module M_2 is α -linear if $\phi(mr) = \phi(m)\alpha(r)$ for $m \in M_1$ and $r \in R$. Let $\mathcal{C}(R, \alpha)$ be the category whose objects are pairs (P, ϕ) where P is a finitely generated projective right R -module and ϕ is an α -linear nilpotent endomorphism of P , and whose morphisms $g: (P_1, \phi_1) \rightarrow (P_2, \phi_2)$ are R -linear homomorphisms $g: P_1 \rightarrow P_2$ such that the following diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{g} & P_2 \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ P_1 & \xrightarrow{g} & P_2 \end{array}$$

is commutative. Let $C'(R, \alpha)$ be the Grothendieck group of the category $\mathcal{C}(R, \alpha)$. The class of an element $(P, \phi) \in \mathcal{C}(R, \alpha)$ in $C'(R, \alpha)$ is denoted by $[P, \phi]$. The class $[R, 0]$ generates a cyclic subgroup $F(R)$ of $C'(R, \alpha)$. Let $C(R, \alpha) = C'(R, \alpha)/F(R)$ and let $\tilde{C}(R, \alpha)$ be the subgroup of $C(R, \alpha)$ generated by $[R^n, \phi]$ for $(R^n, \phi) \in \mathcal{C}(R, \alpha)$. For any abelian group G with an automorphism β , let $G^{\beta} = \{g \mid \beta(g) = g, g \in G\}$, and let $I(\beta) = \{g - \beta(g) \mid g \in G\}$. Then we have:

THEOREM 1 ([4, THEOREM 19] AND [2]) (*Farrell-Hsiang decomposition formula for $K_1 R_{\alpha}[T]$; Bass-Heller-Swan decomposition formula when $\alpha = \text{identity}$*).

$$K_1 R_{\alpha}[T] \cong X \oplus \tilde{C}(R, \alpha) \oplus \tilde{C}(R, \alpha^{-1})$$

and X is given by the following exact sequence

$$0 \rightarrow K_1 R / I(\alpha_*) \rightarrow X \rightarrow (K_0 R)^{\alpha_*} \rightarrow 0,$$

where α_* is the automorphism of $K_1 R$ (resp. $K_0 R$) induced by α .

THEOREM 2 ([4, THEOREM 21] AND [2]). (Farrell-Hsiang decomposition formula for $\text{Wh}(G \times_{\alpha} T)$; Bass-Heller-Swan decomposition formula when $\alpha = \text{identity}$).

$$\text{Wh}(G \times_{\alpha} T) \cong X \oplus \tilde{C}(Z(G), \alpha) \oplus \tilde{C}(Z(G), \alpha^{-1})$$

and X is given by the following exact sequence

$$0 \rightarrow \text{Wh } G / I(\alpha_*) \rightarrow X \rightarrow (\tilde{K}_0 Z(G))^{\alpha_*} \rightarrow 0.$$

Now, we recall that a ring R is right coherent if any homomorphism $f: R^n \rightarrow R^m$ has finitely generated kernel. In [5], Farrell has shown that if R is right regular, then $\tilde{C}(R, \alpha) = 0$ for any automorphism α of R . But, in [3], Choo, Lam and Luft have pointed out that if R is right coherent and has finite right global dimension, then $\tilde{C}(R, \alpha) = 0$ for any automorphism α of R . This result implies, for example, that for a group G with $Z(G)$ right coherent and of finite right global dimension, the exotic summand $\tilde{C}(Z(G), \alpha)$ in the Farrell-Hsiang decomposition formula for $\text{Wh}(G \times_{\alpha} T)$ becomes zero. Moreover, they proved, in particular, that for any free group F and for any abelian group A , $Z(A \times F)$ is right coherent.

Next, let $R(G)$ be the group ring of G over a ring R . We denote the cokernel of the homomorphism $K_1 R \rightarrow K_1 R(G)$ (induced by the inclusion $R \rightarrow R(G)$) by $\tilde{K}_1 R(G)$. The following result is an immediate consequence of [8, Theorem 6.2].

THEOREM 3. Let R be a ring such that $\tilde{C}(R, \text{id}) = 0$ and let G_1 and G_2 be groups. Then $\tilde{K}_1 R(G_1 * G_2) \cong \tilde{K}_1 R(G_1) \oplus \tilde{K}_1 R(G_2)$. In particular, if F is a free group of rank m , we write $F = T_1 * T_2 * \cdots * T_m$ as a free product of m infinite cyclic groups, so that

$$\tilde{K}_1 R(F) \cong \bigoplus_{j=1}^m \tilde{K}_1 R(T_j).$$

PROOF OF LEMMA 1. Let the number of noncyclic factors in D be k . We will prove the lemma by induction on k .

For $k=0$, D is just a free abelian group so that $\text{Wh } D = 0$ and $\tilde{K}_0 Z(D) = 0$ (cf. [2]). This starts the induction.

Now, suppose inductively that the lemma holds for any such group with $k-1$ noncyclic factors. To show $\text{Wh } D = 0$, write $D = D' \times F$ where F is noncyclic, and the number of noncyclic factors in D' is $k-1$. Then for any infinite cyclic group T , $\text{Wh}(D' \times T) = 0$ by induction hypothesis. It follows

from Theorem 2 that

$$(1) \quad \tilde{C}(Z(D'), \text{id}) = 0 \quad \text{and} \quad \tilde{K}_0 Z(D') = 0.$$

Next, suppose that F is of finite rank m and write $F = T_1 * T_2 * \cdots * T_m$ as a free product of m infinite cyclic groups. Since $\tilde{C}(Z(D'), \text{id}) = 0$, it follows from Theorem 3 that

$$(2) \quad \tilde{K}_1 Z(D')(F) \cong \bigoplus_{j=1}^m \tilde{K}_1 Z(D')(T_j).$$

Again, using $\tilde{C}(Z(D'), \text{id}) = 0$, we deduce from Theorem 1 that the sequence $0 \rightarrow K_1 Z(D') \rightarrow K_1 Z(D')(T_j) \rightarrow K_0 Z(D') \rightarrow 0$ is short exact for each j ; i.e., $\tilde{K}_1 Z(D')(T_j) = K_0 Z(D')$ for each j . Hence, it follows from (2) that $\tilde{K}_1 Z(D')(F) \cong K_0 Z(D') \oplus \cdots \oplus K_0 Z(D')$ (m copies); i.e., the sequence

$$(3) \quad 0 \rightarrow K_1 Z(D') \rightarrow K_1 Z(D')(F) \rightarrow K_0 Z(D') \oplus \cdots \oplus K_0 Z(D') \rightarrow 0$$

is short exact. Passing to Whitehead groups, we have (cf. [4])

$$0 \rightarrow \text{Wh } D' \rightarrow \text{Wh}(D' \times F) \rightarrow \tilde{K}_0 Z(D') \oplus \cdots \oplus \tilde{K}_0 Z(D') \rightarrow 0.$$

Hence, by (1) and the induction hypothesis for D' , we have $\text{Wh } D = \text{Wh}(D' \times F) = 0$.

The case when F has infinite rank does not need to worry us since a matrix over $Z(D')(F)$ involves entries which are sums of words involving only a finite number of free generators of F .

Finally, it follows from Theorem 2 and the triviality of $\text{Wh}(D \times T)$ that $\tilde{K}_0 Z(D) = 0$. This completes the proof.

We remark that the first result in Lemma 1 extends that of Bass-Heller-Swan [2] for free abelian groups and of Gersten-Stallings [8] for free groups; and the other extends that of Bass-Heller-Swan [2] for free abelian groups, of Bass [1] for free groups and of Gersten [6] for direct product of a free abelian group and a free group.

PROOF OF THEOREM B. Let k be the number of *noncyclic* factors in D . We prove by induction on k .

For $k=0$, D is just a free abelian group A and so, by Theorem 2, $\text{Wh}(D \times_\alpha T) = 0$ since $\tilde{C}(Z(A), \alpha) = 0$, $\text{Wh } A = 0$ and $\tilde{K}_0 Z(A) = 0$ (cf. [2]).

For $k=1$, D is of the form $A \times F$ with A free abelian of finite rank and F noncyclic. Then, in the Farrell-Hsiang decomposition formula for $\text{Wh}((A \times F) \times_\alpha T)$ (cf. Theorem 2), the terms $\text{Wh}(A \times F)$ and $\tilde{K}_0 Z(A \times F)$ are trivial by Lemma 1. Also, thanks to the coherence property of $Z(A \times F)$ and the fact that $Z(A \times F)$ is of finite right global dimension, $\tilde{C}(Z(A \times F), \alpha) = 0$ (cf. [3]). Hence $\text{Wh}(D \times_\alpha T) = \text{Wh}((A \times F) \times_\alpha T) = 0$.

Now, suppose inductively that the theorem holds for any such group with $k-1$ noncyclic factors. Let $D = H \times F$ with F noncyclic and α fixed

on F while H has $k-1$ noncyclic factors. To show that $\text{Wh}(D \times_{\alpha} T) = 0$, write $D \times_{\alpha} T = D' \times F$ with $D' = H \times_{\alpha} T$. The situation is now completely analogous to Lemma 1 and the same argument there gives $\text{Wh}(D \times_{\alpha} T) = 0$. This completes the proof.

PROOF OF THEOREM A. Consider the group $(D \times_{\alpha} T) \times T_1$, where T and T_1 are infinite cyclic groups. This is just the group $(D \times T_1) \times_{\alpha \times \text{id}_{T_1}} T$. By Theorem B, $\text{Wh}((D \times T_1) \times_{\alpha \times \text{id}_{T_1}} T) = 0$ so that $\text{Wh}((D \times_{\alpha} T) \times T_1) = 0$. The result now follows from Theorem 2. This completes the proof.

Finally, we note that there is a topological application of Theorem B. If M is a closed surface (other than the real projective plane) and $(S^1)^k$ is the k -dimensional torus, then the fundamental group of $M \times (S^1)^k$ is of the form $D \times_{\alpha} T$. Hence, the triviality of $\text{Wh}(D \times_{\alpha} T)$ implies (cf. [7, p. 393]).

COROLLARY. *If N is a differentiable or PL manifold of $\dim \geq 5$ which is h -cobordant to $M \times (S^1)^k$, where M is a closed surface, then N is actually diffeomorphic or PL-homeomorphic to $M \times (S^1)^k$ respectively.*

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