NONINVERTIBLE KNOTS OF CODIMENSION 2

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ABSTRACT. A proof of the noninvertibility of the pretzel knot $(25, -3, 13)$ is given which applies to the knots obtained by repeatedly spinning the pretzel knot.

1. Let JH denote the integral group ring of the infinite cyclic group H . Conjugation in JH is the linear extension of $t \mapsto t^{-1}$, and is denoted by $\alpha \mapsto \bar{\alpha}$.

Let A be an Alexander matrix of a classical knot k ; we may take A to be a square matrix over JH with nonzero determinant. By the work of Blanchfield [1], $\bar{A}'$ is also an Alexander matrix of k , where A' denotes the transpose of A . Let k' denote the knot obtained by inverting k , i.e. by reversing the orientation of S^1 ; then $\bar{A}$, and hence A' , is an Alexander matrix of k' .

In [3], Fox and Smythe describe two invariants of the knot k . If E is the ideal of JH generated by an irreducible factor of the Alexander polynomial $\Delta(t)$, then the row ideal class of A over the ring JH/E is an invariant of k . The same is true of the column class.

Denoting the row class by ρ and the column class by τ , we see that the corresponding row class of k' is τ . Thus for k to be invertible it is necessary that $\rho = \tau$; as $\rho\tau = 1$ in the ideal class semigroup (see [5]), this condition reduces to $\rho^2 = 1$.

2. Now let k be the pretzel knot $(25, -3, 13)$, and take $E = (\Delta(t))$. By [3], an Alexander matrix for this knot is

$$A = \begin{vmatrix} 2t - 1 & -5(t - 1) \\ 11(t - 1) & t - 2 \end{vmatrix}.$$

By taking the product of the ideals generated by the first and second rows, and remarking that $t - 1$ is a unit in JH/E (cf. [3]) we find that ρ^2 is represented by the ideal $(11(2t - 1), 55, 5(t - 2))$. By the argument given in

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[3], to prove that this is not a principal ideal it suffices to show that $J = (55, 11(52 + \sqrt{-211}), 5(107 - \sqrt{-211})/2)$ is not a principal ideal in the ring of integers of the algebraic number field $\mathbb{Q}(\sqrt{-211})$.

The h.c.f. of the norms of these generators is 55, and it is easily checked that every element of J must have norm divisible by 55. Thus if $J = (\alpha)$, the norm of α must be 55. Therefore the only possibilities for α are $(3 \pm \sqrt{-211})/2$: this is a matter of solving the equation $4 \cdot 55 = a^2 + 211b^2$ in integers. Again it is easy to check that $11(52 + \sqrt{-211})$ is not a multiple of $(3 + \sqrt{-211})/2$, nor $5(107 - \sqrt{-211})/2$ of $(3 - \sqrt{-211})/2$.

It follows that J is not a principal ideal, and so $\rho^2 \neq 1$ in the ideal class semigroup.

Therefore k is not invertible.

3. THEOREM 1. *There exist noninvertible n -knots $S^n \subset S^{n+2}$ for all $n \geq 1$.*

PROOF. Let k be the pretzel knot $(25, -3, 13)$; and let k_n be the n -knot obtained by spinning k $(n-1)$ times. Then the group of k_n is isomorphic to that of k , and so the two knots have the same Alexander matrices and hence the same row ideal classes. By the argument above, k_n is non-invertible. $\square$

To emphasize that noninvertibility depends on the module structure rather than on the knot group, we give the following result.

THEOREM 2. *If $1 < q < n/2$, there exists a noninvertible n -knot $S^n \subset S^{n+2}$ with $\pi_i(S^{n+2} - S^n) \cong \pi_i(S^1)$, $i < q$.*

PROOF. By a result of Kervaire [4], there exists such an n -knot with $\pi_q(S^{n+2} - S^n)$ presented as a JH -module by the matrix A given above. $\square$

4. It was pointed out in [2] that a 2-knot with asymmetric Alexander polynomial is noninvertible: by spinning this generalises to $n > 2$. Trotter settled the case $n=1$ in [7]. The method above gives a unified proof for all n , and shows that noninvertibility in higher dimensions need not depend on an asymmetric polynomial.

After writing this paper, I learned that Trotter had previously discovered this method of proving that a 1-knot is noninvertible [6].

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