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COLENGTH OF DERIVATION IDEALS

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ABSTRACT. In this paper, D is a derivation acting on the formal power series ring $K[[t_1, \cdots, t_r]]$ over a field K of characteristic $p\neq 0$. We conjecture that the colengths of the ideals $(D^{p^n}t_1, \cdots, D^{p^n}t_r)$ and $(D^{p^{n-1}}t_1, \cdots, D^{p^{n-1}}t_r)$ are congruent modulo p^n, provided they are finite. We give a proof for the case $r=1$ and any $n\geqq 1$, and for the case $n=1$ and any $r\geqq 1$.

1. **Introduction.** Let $A=K[[t_1, \cdots, t_r]]$ be the ring of formal power series in r-variables over a field K of characteristic $p\neq 0$. The colength of an ideal $\mathfrak{A}$ of A is the dimension of the quotient ring $A/\mathfrak{A}$ viewed as a vector space over K. Given D, a K-derivation of A, we let $\mathfrak{A}(D)$ denote the ideal $(Dt_1, \cdots, Dt_r)$ of A. Since D^p is again a derivation, we have a descending chain of ideals:

$$\mathfrak{A}(D) \supset \mathfrak{A}(D^p) \supset \mathfrak{A}(D^{p^2}) \supset \cdots .$$

CONJECTURE. *Suppose D is a K-derivation of A. Let $i(n)$ denote the colength of the ideal $\mathfrak{A}(D^{p^n})$. Then the congruence $i(n)\equiv i(n-1)$ (modulo p^n) holds for each integer $n\geqq 1$ such that $i(n)<\infty$.*

This paper contains a proof of the conjecture in the one-variable case and a proof of the initial congruence, $i(1)\equiv i(0)$ (modulo p), in the general case.

Suppose that σ is a K-automorphism of A, and let 1 denote the identity automorphism of A. We may view the operator $L=\sigma-1$ as a "twisted derivation" in the sense that $L(xy)=xL(y)+\sigma(y)L(x)$.

Consider the analog of the above conjecture obtained by substituting L for D. In the one-variable case, the congruences of this analog are implied by the Hasse-Arf theorem as generalized by Sen [2]. For general A and σ of finite order, the analog would follow from an affirmative answer to a question asked by Serre [4, p. 418].

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2. **The one-variable case, $A=K[[t]]$.** For $\alpha \in A$, let α_t denote the formal partial derivative of α with respect to t. Thus $D\alpha=\alpha_t Dt$ for all derivations D.

LEMMA. *Let D be a K-derivation of A. There exists a "constant" $c \in K[[t^p]]$, such that $D^{s+p-1}t=cD^s t$, for $s \geqq 1$.*

PROOF. Let $Dt=f$, which we may assume is not zero. As $\mathfrak{A}(D^p) \subset \mathfrak{A}(D)$, we may write $D^p t=cDt$ for some $c \in A$. Now D and D^p are derivations, so

$$D^{p+1}t = D(D^p t) = D(cf) = cDf + fDc$$

and

$$D^{p+1}t = D^p(Dt) = D^p f = f_t D^p t = f_t cf = cDf.$$

It follows that $fDc=0$. Hence, $c \in K[[t^p]]$. Now for $s \geqq 1$, we have

$$D^{s+p-1}t = D^{s-1}(D^p t) = D^{s-1}(cDt) = cD^s t. \quad \square$$

Let v denote the valuation of A corresponding to the ideal (t) and written additively. The colength of an ideal is then v of a generator of that ideal. The lemma implies that $D^{m(p-1)+1}t=c^m Dt$ for $m \geqq 1$. Taking $m=(p^n-1)/(p-1)$, we find that the colength of $\mathfrak{A}(D^{p^n})$ is given by

$$i(n) = v(D^{p^n}t) = (p^{n-1} + \cdots + 1)v(c) + v(Dt).$$

Similarly

$$i(n-1) = (p^{n-2} + \cdots + 1)v(c) + v(Dt).$$

Subtracting, we find that $i(n)-i(n-1)=p^{n-1}v(c)$. But p divides $v(c)$ because $c \in K[[t^p]]$. Hence, $i(n) \equiv i(n-1)$ (modulo p^n) and the conjecture is true in the one-variable case.

3. **Partial result when $A=K[[t_1, \cdots, t_r]]$.** The congruence $i(1) \equiv i(0)$ (modulo p) may be obtained with the aid of the *residue symbol.* (See Hartshorne [**1**, pp. 195–199].)

When ω is an r-differential form on A over K and the ideal $(f_1, \cdots, f_r)$ has finite colength, the residue symbol

$$\begin{bmatrix} \omega \\ f_1, \cdots, f_r \end{bmatrix}$$

taking values in K is defined. We make use of the following properties, assuming in each case that the conditions for the existence of the residue symbol are fulfilled:

(a) Let $\mathfrak{A}=(f_1, \cdots, f_r)$. Then

$$\begin{bmatrix} df_1 \wedge \cdots \wedge df_r \\ f_1, \cdots, f_r \end{bmatrix} = \text{colength}(\mathfrak{A}) \cdot 1_K.$$

(b) If $\alpha \in \mathfrak{A}=(f_1, \cdots, f_r)$, then

$$\begin{bmatrix} \alpha\omega \\ f_1, \cdots, f_r \end{bmatrix} = 0.$$

(c) If the elements $g_i=\sum_{j=1}^r \alpha_{ij}f_j$, for $i=1, \cdots, r$, generate another ideal of finite colength, then

$$\begin{bmatrix} \det(\alpha_{ij})\omega \\ g_1, \cdots, g_r \end{bmatrix} = \begin{bmatrix} \omega \\ f_1, \cdots, f_r \end{bmatrix}.$$

LEMMA. *Suppose that* $(f_1, \cdots, f_r)$ *has finite colength and* α_{ij}, β_{ij} *are elements of* A *such that*

$$g_i = \sum_{j=1}^r \alpha_{ij}f_j = \sum_{j=1}^r \beta_{ij}f_j, \qquad i = 1, \cdots, r.$$

Then $\det(\alpha_{ij})-\det(\beta_{ij}) \in (g_1, \cdots, g_r)$.

PROOF. To fix notation, let the Koszul complex $K_*(f_1, \cdots, f_r)=K_*(f)$ be given as follows (see Serre [3, Chapter IV]): $K_m(f)=\bigwedge^m (A^r)$, the mth exterior product of A^r. Let $e_1, \cdots, e_r$ denote the generators of $K_1(f)$. The boundary map $\delta_m: K_m(f)\to K_{m-1}(f)$ is given by

$$e_{i_1} \wedge \cdots \wedge e_{i_m} \to \sum_{j=1}^m (-1)^{j+1} f_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_m}.$$

Let $E_1, \cdots, E_r$ denote the generators of $K_1(g)$. Define a map of A-algebras $S_*: K_*(g)\to K_*(f)$ by the following action on generators: $E_i\to\sum_{j=1}^r \alpha_{ij}e_j$. Because $g_i=\sum_{j=1}^r \alpha_{ij}f_j$, the following diagram is commutative:

$$\begin{array}{ccc} K_1(g) & \xrightarrow{\delta} & K_0(g) = A \\ {\scriptstyle S_1}\downarrow & & \downarrow{\scriptstyle \text{identity}} \\ K_1(f) & \xrightarrow{\delta} & K_0(f) = A \end{array}$$

and S commutes with the boundary map δ. In particular S_r is multiplication by $\det(\alpha_{ij})$ in dimension r.

Similarly, we define $T: K_*(g)\to K_*(f)$ by replacing α_{ij} by β_{ij} above.

$K_*(g)$ is free. $K_*(f)$ is exact in positive dimensions because the ideal $(f_1, \cdots, f_r)$ has finite colength [3, p. IV-5]. Since $S_0=T_0=$identity in dimension zero, there is a standard way (Spanier [5, p. 165]) to construct $h_m: K_m(g)\to K_{m+1}(f)$ such that $\delta_{m+1}h_m+h_{m-1}\delta_m=S_m-T_m$. In dimension r, we find that

$$\begin{aligned}[\det(\alpha_{ij}) - \det(\beta_{ij})]e_1 \wedge \cdots \wedge e_r &= (S_r - T_r)E_1 \wedge \cdots \wedge E_r \\ &= h_{r-1}\delta_r E_1 \wedge \cdots \wedge E_r \\ &= \gamma e_1 \wedge \cdots \wedge e_r\end{aligned}$$

where $\gamma \in (g_1, \cdots, g_r)$.

It follows that $\det(\alpha_{ij}) - \det(\beta_{ij}) \in (g_1, \cdots, g_r)$ and the lemma is proved.

THEOREM. $i(1) \equiv i(0)$ (modulo p).

PROOF. Let $\partial_i \alpha$ denote the formal partial derivative of α with respect to t_i. Thus $D\alpha = \sum_1^r \partial_i \alpha Dt_i$.

Let $Dt_i = f_i$, and $D^p t_i = F_i$. As $(F_1, \cdots, F_r) \subset (f_1, \cdots, f_r)$ we may write

$$F_i = \sum_{j=1}^{r} c_{ij} f_j, \qquad i = 1, \cdots, r. \tag{1}$$

Using properties (a) and (c) of the residue symbol, we find that

$$i(0) \cdot 1_K = \begin{bmatrix} \det(c_{ij})\, df_1 \wedge \cdots \wedge df_r \\ F_1, \cdots, F_r \end{bmatrix} = \begin{bmatrix} \det(c_{ij}) \det(\partial_i f_j)\, dt_1 \wedge \cdots \wedge dt_r \\ F_1, \cdots, F_r \end{bmatrix}$$

and

$$i(1) \cdot 1_K = \begin{bmatrix} dF_1 \wedge \cdots \wedge dF_r \\ F_1, \cdots, F_r \end{bmatrix} = \begin{bmatrix} \det(\partial_i F_j)\, dt_1 \wedge \cdots \wedge dt_r \\ F_1, \cdots, F_r \end{bmatrix}.$$

Subtracting, we find that

$$[i(1) - i(0)] \cdot 1_K = \begin{bmatrix} \gamma\, dt_1 \wedge \cdots \wedge dt_r \\ F_1, \cdots, F_r \end{bmatrix}$$

where $\gamma = \det(\partial_i F_j) - \det(c_{ij}) \det(\partial_i f_j)$.

We now obtain two expressions for $g_i = D^{p+1} t_i$ by using the fact that both D and D^p are derivations.

$$g_i = D(D^p t_i) = D(F_i) = \sum_{j=1}^{r} (\partial_j F_i) f_j$$

and

$$g_i = D^p(Dt_i) = D^p(f_i) = \sum_{j=1}^{r} (\partial_j f_i) F_j.$$

Substitute the expression in (1) for F_j above and apply the lemma of this section to show that $\gamma = \det(\partial_i F_j) - \det(c_{ij}) \det(\partial_i f_j)$ is an element of the ideal $(g_1, \cdots, g_r) \subset (F_1, \cdots F_r)$.

Now, by property (b) of the residue symbol, $[i(1) - i(0)] \cdot 1_K = 0$. Hence the theorem is proved. □

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