ON SEMIPRIME P.I. RINGS

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ABSTRACT. The main results proved in this paper are that if R is a semiprime ring satisfying a polynomial identity then (1) the maximal right quotient ring of R is also P.I. and (2) every essential one-sided ideal of R contains an essential two-sided ideal of R .

The primary aim of this paper is to settle in the affirmative two recent conjectures of J. W. Fisher [1]: If R is a semiprime ring satisfying a polynomial identity then

- (I) the maximal right quotient ring of R satisfies a polynomial identity,
- (II) the maximal right quotient ring of R coincides with the maximal left quotient ring of R .

The main tool we use is a result (Theorem A, below) just recently proved by L. Rowen [4, Theorem 2], which in turn depends heavily on a fundamental theorem on central polynomials in matrix rings due to E. Formanek [2].

Let R be a ring, not necessarily having a unit element. If S is a subset of R , we let $r(S)$ denote $\{x \in R \mid Sx=0\}$ and $l(S)=\{x \in R \mid xS=0\}$. A right (left, two-sided) ideal J of R is essential if, for any right (left, two-sided) ideal K , $J \cap K=0$ implies $K=0$. A ring is semiprime if it has no nonzero nilpotent ideals. It is clear that in a semiprime ring a two-sided ideal U is essential if and only if $r(U)=0$ or $l(U)=0$, and an essential two-sided ideal is also essential as a one-sided ideal. Also in a semiprime ring it is easy to see that if J is an essential right ideal then $r(J)=0$.

The set $Z(R)=\{x \in R \mid xJ=0 \text{ for some essential right ideal } J\}$ is a two-sided ideal of R , called the right singular ideal of R . Similarly the left singular ideal $Z'(R)$ is defined. As we shall be concerned with rings R for which $Z(R)=0$ (see Theorem 1) the maximal right quotient ring Q of R may be characterized by the following properties:

- (a) R is a subring of Q .
- (b) If $f \in \text{Hom}_R(J, R)$, where J is an essential right ideal of R , then there exists $q \in Q$ such that $qx=f(x)$ for all $x \in J$.

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(c) If $q \in Q$, there is an essential right ideal J of R such that $qJ \subseteq R$.

(d) For all $q \in Q$, $q=0$ if and only if $qJ=0$ for some essential right ideal J in R .

It is well known that Q is regular in the sense of von Neumann.

As in [1] a ring R is said to satisfy a polynomial identity if there exists a (homogeneous multilinear) polynomial

$$f(x_1, x_2, \dots, x_n) = \sum_i \omega_i x_{i_1} x_{i_2} \cdots x_{i_n}$$

in noncommuting indeterminates $\{x_i\}$ (where i ranges over the symmetric group S_n and ω_i lies in the centroid of R) such that $f(r_1, r_2, \dots, r_n)=0$ for all $r_1, r_2, \dots, r_n \in R$ and the kernel of $\omega_1=0$. Such a ring is called a P.I. ring. We remark that a more general definition of polynomial identity could have been made which does not require f to be homogeneous multilinear, but the usual linearization process could be used to replace f by a homogeneous multilinear polynomial of degree not exceeding the degree of f . On the other hand, in case R is a semiprime P.I. ring, it is well known that R in fact satisfies the so-called standard identity $\sum \pm x_{i_1} x_{i_2} \cdots x_{i_n} = 0$.

The key theorem which lies behind the results in this paper is due to L. Rowen [4, Theorem 2] and we quote it as

THEOREM A (ROWEN). *Let R be a semiprime P.I. ring with center C and let U be a nonzero ideal of R . Then $U \cap C \neq 0$.*

It is important to observe that the assumption made in [4] that R has a unit element is superfluous—the same proof may still be used.

LEMMA 1. *Let R be a semiprime ring with center C and let J be a right ideal of R . Then the center of the ring J is equal to $J \cap C$.*

PROOF. Let a lie in the center of J and let $x, r \in R$. Then

$$\begin{aligned} (ax - xa)r(ax - xa) &= (axr)ax - (axrx)a + x(arx)a - x(ar)ax \\ &= a(axr)x - a(axrx) + xa(arx) - xa(ar)x \\ &= 0. \end{aligned}$$

Since R is semiprime, $ax - xa = 0$, i.e., $a \in C$.

Henceforth R will denote a semiprime P.I. ring with center C . As a corollary to Theorem A we obtain immediately the following result due to Fisher [1, Theorem 1].

THEOREM 1 (FISHER). $Z(R)=0=Z'(R)$.

PROOF. If $Z(R) \neq 0$, pick $\lambda \neq 0 \in Z(R) \cap C$ by Theorem A. Then, for some essential right ideal J , $J\lambda = \lambda J = 0$. Since R is semiprime, this contradicts $r(J)=0$.

LEMMA 2. *If J is an essential right (left) ideal of R , then J is itself a semiprime P.I. ring.*

PROOF. Suppose $A^2=0$, where A is an ideal of J . Then AJ is a right ideal of R and $(AJ)^2 \subseteq A^2=0$. Hence $AJ=0$ since R is semiprime. But then $A=0$ by Theorem 1.

We are now in a position to settle conjecture I.

THEOREM 2. *If R is a semiprime P.I. ring (satisfying the identity f) then the maximal right quotient ring Q of R satisfies this same polynomial identity.*

PROOF. Write $f(x_1, x_2, \dots, x_n) = \sum_i \omega_i x_{i_1} x_{i_2} \cdots x_{i_n}$, pick $q_1, q_2, \dots, q_n \in Q$, and set $q = f(q_1, q_2, \dots, q_n)$. By taking finite intersections there exists an essential right ideal J of R such that $q_1 J, q_2 J, \dots, q_n J, qJ$ are all contained in R . Let $a \in J$ and write $qa = b \in R$. Suppose $b \neq 0$. Then $U = RbR \cap J \neq 0$, since J is essential, and U is a two-sided ideal of the ring J . By Lemma 2, J is a semiprime P.I. ring, and so we can apply Theorem A to J to conclude that U contains a nonzero element λ in the center of J . But then by Lemma 1 $\lambda \in C$. Therefore

$$\begin{aligned} b\lambda^n &= qa\lambda^n = \sum_i \omega_i q_{i_1} q_{i_2} \cdots q_{i_n} a\lambda^n \\ &= \left\{ \sum \omega_i (q_{i_1} \lambda) (q_{i_2} \lambda) \cdots (q_{i_n} \lambda) \right\} a = 0, \end{aligned}$$

since each $q_i \lambda \in R$. But this says that $\lambda^{n+1} \in \lambda^n RbR = R(\lambda^n b)R = 0$, a contradiction to $\lambda \neq 0$. Thus $qJ=0$ and so $q=0$.

The next theorem will be instrumental in proving conjecture II.

THEOREM 3. *Let J be any right (left) ideal of R . Then either $l(J) \neq 0$ ($r(J) \neq 0$) or J contains a two-sided essential ideal of R .*

PROOF. By symmetry we may assume J is a right ideal. Suppose $l(J)=0$. If A is an ideal of J such that $A^2=0$, then $(AJ)^2 \subseteq A^2=0$. Since R is semiprime we have $AJ=0$, i.e., $A \subseteq l(J)$. Therefore $A=0$ and so J is a semiprime ring. Similarly $[r(J)J]^2=0$ and $r(J)J=0$, forcing $r(J)=0$.

Let U be the largest two-sided ideal of R which is contained in J . Suppose U is not an essential ideal of R . Then there is a nonzero ideal V such that $UV=0$. $V \cap J \neq 0$, otherwise $JV=0$, leading to $VJ=0$, contradicting $l(J)=0$. By Theorem A, applied to the semiprime ring J (and also using Lemma 1), $V \cap J \cap C \neq 0$. Picking $0 \neq \lambda \in V \cap J \cap C$, we obtain an ideal $U + \lambda R$ of R lying in J and properly containing U .

Theorem 1 and Theorem 3 together immediately imply.

THEOREM 4. *Let J be an essential one-sided ideal of R . Then J contains an essential two-sided ideal of R .*

The following theorem answers conjecture II.

THEOREM 5. *Q coincides with the maximal left quotient ring of R .*

PROOF. Utumi has shown [5, p. 145, Theorem 3.3] that if both singular ideals of R are zero, then the maximal right and left quotient rings coincide if and only if for every nonessential right (left) ideal J of R the left (right) annihilator of J is nonzero. But Theorem 3 precisely assures these conditions.

We recall from Lemma 1 that if J is a right ideal of R then $J \cap C$ is in fact the center of the ring J and is an ideal of C .

THEOREM 6. *A right ideal J is essential in R if and only if $J \cap C$ is essential in C .*

PROOF. If J is essential in R then by Theorem 4 J contains an essential 2-sided ideal U of R . If $\lambda \neq 0 \in C$, λU is a nonzero ideal of R since U is essential. By Theorem A, $0 \neq \lambda u \in C$ for some $u \in U$. Suppose $\lambda(J \cap C) = 0$. Then $\lambda(U \cap C) = 0$, and in particular $(\lambda u)^2 = u\lambda(\lambda u) = 0$, a contradiction. Hence $\lambda(J \cap C) \neq 0$ and so $J \cap C$ is essential in C .

Conversely, if $J \cap C$ is essential in C , let U be the ideal of R generated by $J \cap C$. Suppose $r(U) \neq 0$. By Theorem A pick $0 \neq \lambda \in r(U) \cap C$. But then $(J \cap C)\lambda = 0$, a contradiction and so $r(U) = 0$. Thus U and hence J (which contains U) is essential in R .

Finally, we reprove some results due to Herstein and Small [3, p. 328, Theorems 2 and 3].

THEOREM 7. *If $l(a) = 0$, then a is regular, aR and Ra are essential right and left ideals, and a is invertible in Q .*

PROOF. By Theorem 3, aR contains an essential ideal U of R and thus aR is essential. Since Q is (von Neumann) regular there exists $q \in Q$ such that $aq a = a$. Thus $(aq - 1)aR = 0$ and so $aq = 1$, since aR is essential. Suppose $qa \neq 1$. It is easily checked that $e_{ij} = q^i(1 - qa)a^j$, $i, j = 1, 2, \dots$, is an infinite set of matrix units in Q . But this cannot occur, since Q is a P.I. algebra by Theorem 2. Hence $qa = 1$, which implies that a is regular and that Ra is an essential left ideal of R .

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The proof of Theorem 3 was somewhat shortened following a suggestion by S. Steinberg. The latter also indicated that a more self-contained proof could be given for Theorem 5, without recourse to Utumi's paper but making use of Theorem 6. An error in our original proof of Theorem 7 was rectified by E. P. Armendariz. Finally we are indebted to the National Science Foundation, which partially supported the research involved in this paper (GP-12090).

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