CHARACTERIZATION OF CERTAIN GROUPOIDS AND LOOPS

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ABSTRACT. Groupoids (loops) possessing the left inverse property (lip) (also the right inverse property) and crossed-inverse (CI) loops are characterized as subvarieties of the variety of groupoids satisfying a single identity.

1. It is known that groups, Abelian groups, loops with the inverse property and the weak inverse property loops, can be defined by means of a single identity as a subvariety of the variety of groupoids [3], [5], [6], [4]. In this paper, we give characterizations for (i) groupoids (loops) with identity satisfying the left inverse property (lip) (also the right inverse property), (ii) crossed-inverse (CI) loops, as subvarieties of groupoids satisfying certain conditions.

A groupoid $G(\cdot)$ is said to have the left inverse property (lip), if, for every $x \in G$, there is at least one $a \in G$ such that $a \cdot (xy) = y$ for all $y \in G$. Consequently, for given $c, d \in G$, there is a unique $x \in G$ such that $cx = d$ holds. Similarly, $G(\cdot)$ possesses the right inverse property (rip) if, for each $x \in G$, there is at least one $b \in G$ such that $(yx) \cdot b = y$ for every $y \in G$. If $G(\cdot)$ has both lip and rip, then G is said to have the inverse property and in this case $G(\cdot)$ is a quasigroup (see [2, p. 111]).

A groupoid $G(\cdot)$ in which the equation $ya = b$ has a unique solution for y , for every given pair $a, b \in G$ is called a right quasigroup. Similarly, if, for every pair $a, b \in G$, there is a unique x such that $ax = b$ holds in $G(\cdot)$, then the groupoid is called a left quasigroup.

Loops $G(\cdot)$ in which the equivalent identities $(xy) \cdot \rho(x) = y$ or $\lambda(x) \cdot (yx) = y$ hold for all $x, y \in G$ are called crossed-inverse (CI) loops, where $\rho(\lambda)$ is the right (left) inverse operator (see [1], [2]).

2. Let $G(\cdot)$ be a groupoid. We say that $G(\cdot)$ is an iso-lip (rip) groupoid (or loop) provided there is a lip (rip) groupoid (or loop) $G(*)$ with unity which is a principal isotope of $G(\cdot)$ such that $\cdot$ and $*$ are connected by the relation $x * y = x \cdot \lambda(y)$ ($x * y = \rho(x) \cdot y$) for all $x, y \in G$, where $\lambda(\rho)$ is the left (right) inverse operator of $G(*)$.

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THEOREM 1. *A necessary and sufficient condition that a groupoid $G(\cdot)$ is an iso-lip groupoid is that the following identity,*

$$(1) \quad (vv) \cdot [\{(tt)x \cdot (uu)(xy)\} \cdot (ww)] = y,$$

or,

$$(1') \quad [(vv) \cdot \{(tt)x \cdot (uu)(xy)\}] \cdot (ww) = y,$$

holds for all $x, y, t, u, v, w \in G$.

PROOF. We will consider only (1). The proof for (1') is exactly the same as that of (1). First, we will show that $(\cdot)$ is left cancellative. Let $a \cdot s = a \cdot r$. Setting in (1), first $x=a, y=s$ and then $x=a, y=r$ and using $a \cdot s = a \cdot r$, we get $s=r$. Thus $(\cdot)$ is left cancellative.

Keeping t, u, v, x and y fixed in (1) and using the left cancellativity of $(\cdot)$, we obtain,

$$(2) \quad w \cdot w = \text{constant} = e(\text{say}), \text{ for all } w \in G.$$

Now (1) becomes

$$(3) \quad e \cdot [\{(ex) \cdot (e \cdot (xy))\} \cdot e] = y, \text{ for all } x, y \in G.$$

In (3), $y=e$ gives, by repeated use of (2) and the left cancellativity of $(\cdot)$,

$$(4) \quad xe = x, \text{ for all } x \in G.$$

Now (3) can be rewritten as

$$(5) \quad e \cdot [(ex) \cdot (e \cdot (xy))] = y, \text{ for all } x, y \in G.$$

Put $y=x$ in (5) to get, using (4), that

$$(6) \quad e \cdot (ex) = x, \text{ for all } x \in G.$$

Define the mapping $\lambda: G \rightarrow G$, such that

$$(7) \quad \lambda(x) = e \cdot x, \text{ for all } x \in G.$$

Then by (6) and the left cancellativity of $(\cdot)$, it follows that λ is a permutation of G .

Now define $G(*)$ as follows:

$$(8) \quad x * y = x \cdot \lambda(y), \text{ for all } x, y \in G.$$

From (8), it is true that $G(*)$ is a principal isotope of $G(\cdot)$ and that $G(*)$ is a groupoid.

Further, using (6), (7) and (8), (5) becomes $\lambda[\lambda(x) \cdot \lambda(xy)] = y$, that is,

$\lambda(x) * [x * \lambda(y)] = \lambda(y)$, that is,

$$(9) \quad \lambda(x) * (x * y) = y, \text{ for all } x, y \in G,$$

which shows that $G(*)$ satisfies the lip.

Also, (8), (6) and (4) imply

$$(10) \quad x * e = x = e * x, \text{ for all } x \in G.$$

Hence $G(\cdot)$ is an iso-lip groupoid.

Conversely, let $G(\cdot)$ be an iso-lip groupoid and let $G(*)$ be the corresponding lip groupoid with unity e such that $(*)$ and $(\cdot)$ are connected by

$$(8) \quad x * y = x \cdot \lambda(y), \text{ for all } x, y \in G,$$

where λ is the left inverse operator of $G(*)$. It is easy to see that $\lambda(x)$ is unique and $x * \lambda(x) = e = \lambda(x) * x$, for every $x \in G$. Thus, $\lambda^2(x) = x$, for every $x \in G$, showing thereby that λ is a permutation of G . From (8), we get

$$(11) \quad x \cdot y = x * \lambda(y).$$

So, $x \cdot x = x * \lambda(x) = e$, for all $x \in G$, $e \cdot y = \lambda(y)$ and $y \cdot e = y * e = y$, for all $y \in G$. Hence (2), (4) and (7) hold true. Use (8), (11), (2), (4) and (7) in (9) to get (1). This completes the proof of this theorem.

COROLLARY 1. *Let $G(\cdot)$ be a right quasigroup in which (1) is true. Then $G(\cdot)$ is an iso-lip loop.*

PROOF. By Theorem 1, $G(\cdot)$ is an iso-lip groupoid. Consequently $a * x = b$ is solvable uniquely for x in $G(*)$, the associated lip groupoid with unity. As $G(\cdot)$ is a right quasigroup, $y \cdot a = b$ is uniquely solvable for y in $G(\cdot)$. This leads to the solvability of $y * a = b$ for y in $G(\cdot)$ showing, thereby, that $G(*)$ is a lip loop. Thus $G(\cdot)$ is an iso-lip loop.

Now, we merely state the corresponding results for rip.

THEOREM 2. *A groupoid $G(\cdot)$ is an iso-rip groupoid if, and only if,*

$$(12) \quad (ww) \cdot [\{(yx \cdot uu) \cdot (x \cdot tt)\} \cdot (vv)] = y$$

or

$$(12') \quad [(ww) \cdot \{(yx \cdot uu) \cdot (x \cdot tt)\}] \cdot (vv) = y$$

is true for all $x, y, t, u, v, w \in G$.

COROLLARY 2. *Let $G(\cdot)$ be a left quasigroup in which (12) is true. Then $G(\cdot)$ is an iso-rip loop.*

3. Let $G(\cdot)$ be a groupoid. Then $G(\cdot)$ is called an iso-CI loop, if there is a CI loop $G(*)$ which is a principal isotope of $G(\cdot)$ such that $\cdot$ and

$*$ are connected either by $x * y = \rho(x) \cdot y$ or by $x * y = x \cdot \lambda(y)$ where λ is the left inverse operator of $G(*)$ and $\rho = \lambda^{-1}$, is the right inverse mapping.

THEOREM 3. *Let $G(\cdot)$ be a right quasigroup. Then $G(\cdot)$ is an iso-CI loop, provided*

$$(13) \quad x \cdot (yx) = y \cdot (tx),$$

holds for all $x, y, t \in G$ and conversely if $G(\cdot)$ is an iso-CI loop, then (13) holds for all $x, y, t \in G$.

PROOF. Let $a \cdot r = a \cdot s$. Then, first with $y=a, x=r$ in (13) and then with $y=a, x=s$ in (13), we get $r \cdot (ar) = a \cdot (tr) = s \cdot (as) = s \cdot (ar)$. Since $G(\cdot)$ is a right quasigroup, we obtain $r=s$, that is, the left cancellativity of $(\cdot)$.

Then from (13), by keeping x and y fixed and using the left cancellativity we have

$$(14) \quad t \cdot t = \text{constant} = e(\text{say}), \text{ for every } t \in G.$$

So, (13) becomes

$$(15) \quad x \cdot (yx) = ye, \text{ for all } x, y \in G.$$

Since $G(\cdot)$ is a right quasigroup, from (15) we see that $ax=b$ implies $ae=xb$ and is then uniquely solvable for x . Thus $G(\cdot)$ is a quasigroup. With $y=e$, (15) gives, using (14), that $x \cdot (ex)=e$, that is,

$$(16) \quad ex = x, \text{ for all } x \in G.$$

Now, consider the mapping $\lambda: G \rightarrow G$, such that

$$(17) \quad \lambda(x) = xe, \text{ for all } x \in G.$$

Then λ is a permutation of G .

Now, define $G(*)$ as follows:

$$(18) \quad x * y = \rho(x) \cdot y, \text{ for all } x, y \in G,$$

where $\rho = \lambda^{-1}$. Then $G(*)$ is a principal isotope of $G(\cdot)$ and $G(*)$ is a quasigroup.

$x=e$ in (18) with (16) gives $e * y = e \cdot y = y$, for all $y \in G$.

$y=e$ in (18) with (17) gives $x * e = \rho(x) \cdot e = x$, for all $x \in G$.

Therefore, e is the identity of $G(*)$ and hence $G(*)$ is a loop.

From (18), we have

$$(19) \quad x \cdot y = \lambda(x) * y, \text{ for } x, y \in G.$$

Making use of (17) and (19), (15) becomes, $x \cdot (yx) = \lambda(y)$, that is,

$x \cdot [\lambda(y) * x] = \lambda(y)$, that is,

$$(20) \quad \lambda(x) * (y * x) = y, \text{ for all } x, y \in G.$$

Hence $G(*)$ is a CI loop.

Conversely, let the right quasigroup $G(\cdot)$ be an iso-CI loop. Let $G(*)$ be the corresponding CI loop such that $(*)$ and $(\cdot)$ are connected by (18) or (19), where λ and ρ are the left and right inverse operators of $G(*)$.

Then (20) holds good in $G(*)$. Now, $y=e$ in (20) gives $\lambda(x) * x=e$. Thus from (19) with $y=x$, we obtain (14).

Also, $y=e$ in (19) gives (17). It is now easy to see from (20), that (13) is true.

Thus the proof of this theorem is complete.

THEOREM 4. *Let $G(\cdot)$ be a left quasigroup. Then $G(\cdot)$ is an iso-CI loop, if and only if*

$$(21) \quad xy \cdot x = (tx) \cdot x$$

is satisfied for all $x, y, t \in G$.

The proof of Theorem 4 is similar to that of Theorem 3.

REMARK. Whereas the isotopy $x * y = \rho(x) \cdot y$ is taken into account in Theorem 3, the isotopy $x * y = x \cdot \lambda(y)$ is to be used in Theorem 4.

A loop $G(\cdot)$ with identity e is said to be of exponent 2 if $x \cdot x = e$ for all $x \in G$.

THEOREM 5. *A groupoid $G(\cdot)$ is a CI loop of exponent 2 if and only if*

$$(22) \quad x \cdot (yx \cdot uu) = y, \text{ for all } x, y, u \in G.$$

PROOF. We will show that under the assumption of (22), $(\cdot)$ is both right and left cancellative.

Let $va=wa$. Then in (22), with $x=a$, $y=v$ first and then $x=a$, $y=w$, we get $v=w$. So, $(\cdot)$ is right cancellative. Let $av=aw$. Setting $x=w$, $y=a$ and $x=v$, $y=a$ in (22), we obtain $w \cdot [aw \cdot uu] = a = v \cdot [av \cdot uu]$, from which and the right cancellativity of $(\cdot)$, we get the left cancellativity of $(\cdot)$.

Changing only u in (22) and using the cancellativities of $(\cdot)$, we have

$$(23) \quad uu = \text{constant} = e(\text{say}), \text{ for all } u \in G.$$

Now, (22) becomes

$$(24) \quad x \cdot (yx \cdot e) = y, \text{ for all } x, y \in G.$$

Put $y=x$ in (24) to get, with the use of (23), that

$$(25) \quad xe = x, \text{ for all } x \in G,$$

so that (24) takes the form,

$$(26) \quad x \cdot yx = y, \text{ for all } x, y \in G.$$

From (25) and (26) with $x=e$, we get

$$(27) \quad ey = y, \text{ for all } y \in G,$$

showing thereby that e is the identity of $G(\cdot)$. By (23), it follows that $G(\cdot)$ is of exponent 2 and by (26) it follows that $G(\cdot)$ is a CI loop. The converse is trivial. This completes the proof of this theorem.

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REFERENCES

1. R. Artzy, *Crossed-inverse and related loops*, Trans. Amer. Math. Soc. **91** (1959), 480–492. MR **21** #5688.
2. R. H. Bruck, *A survey of binary systems*, Springer-Verlag, Berlin and New York, 1966.
3. G. Higmann and B. H. Neumann, *Groups as groupoids with one law*, Publ. Math. Debrecen **2** (1952), 215–221. MR **15**, 284.
4. Pl. Kannappan, *On weak inverse property loops*, J. London Math. Soc. **5** (1972), 298–302.
5. ———, *Groupoids and groups*, Jber. Deutsch. Math. Verein. (to appear).
6. R. Padmanabhan, *Inverse loops as groupoids with one law*, J. London Math. Soc. (2) **1** (1969), 203–206. MR **39** #5744.

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