PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, October 1973

PROBABILITY MEASURES ON SEMIGROUPS

PETER GERL

ABSTRACT. Let S be a discrete semigroup, P a probability measure on S and $s \in S$ with $\limsup_n (P^{(n)}(s))^{1/n} = 1$. We study limit theorems for the convolution powers $P^{(n)}$ of P implied by the above property and further the class of all semigroups with this property. Theorem 3 relates this class of semigroups to left amenable semigroups.

1. **Introduction.** Let S be a discrete semigroup and P a *probability measure* on S, that is a real valued function on S with $P(s) \geqq 0$ for all $s \in S$ and $\sum_{s\in S} P(s) = 1$. Kesten ([4], [5]) characterized amenable groups by means of the asymptotic behavior of convolution powers of symmetric probability measures defined on the group. A more precise information for the asymptotic behavior was obtained in [2] and [3] for symmetric probability measures on a discrete amenable group. In what follows we will derive similar theorems for probability measures on discrete semigroups.

Let S be a discrete semigroup, P a probability measure on S. Then $\operatorname{Supp} P = \{s/P(s) > 0\}$ denotes the *support* of P. To say Supp P *generates* S means: $S = \bigcup_{n=1}^{\infty} (\operatorname{Supp} P)^n$.

For probability measures P, Q on S define their *convolution* $P * Q$ by

$$P * Q(s) = \sum_{s_1 s_2 = s} P(s_1)Q(s_2)$$

(the summation is to be extended over all representations of s as a product of two elements s_1, s_2 of S). $P * Q$ is again a probability measure and $\operatorname{Supp} P * Q = (\operatorname{Supp} P) \cdot (\operatorname{Supp} Q)$. We often write $P^{(1)} = P, P^{(n)} = P * P^{(n-1)}$.

Kesten obtained the following characterization of discrete amenable groups:

Let G be a discrete group with unit element e, P a symmetric probability measure on G ($P(g) = P(g^{-1})$ for all g in G) such that G is generated by

---

Received by the editors January 2, 1973.

AMS (MOS) subject classifications (1970). Primary 43A05, 43A07; Secondary 20M10, 60B15, 60F99.

Key words and phrases. Semigroup, probability measure, convolution, limit theorems, amenable semigroup.

Supp P; *then*

$$G \text{ is amenable} \Leftrightarrow P[e] = \limsup_{n\to\infty} (P^{(n)}(e))^{1/n} = 1.$$

2. **Limit theorems.** Let S be a discrete semigroup; if there exists a probability measure P on S with the properties

(1) Supp P generates S, and

(2) there exists an $s \in S$ with $P[s] = \limsup_{n\to\infty} (P^{(n)}(s))^{1/n} = 1$ then we call S an *A-semigroup* or, if we want to specify P, (S, P) an *A-pair*.

Let (S, P) be an A-pair with $P[s]=1$ for some $s \in S$. Let $s' \in S$; since Supp P generates S there exists a natural number k with $P^{(k)}(s')>0$. Then

$$P^{(n+k)}(ss') = \sum_{s_1 s_2 = ss'} P^{(n)}(s_1)P^{(k)}(s_2) \geqq P^{(n)}(s)P^{(k)}(s')$$

and (k is fixed)

$$\begin{aligned} 1 \geqq P[ss'] &= \limsup_n (P^{(n+k)}(ss'))^{1/(n+k)} \\ &\geqq \limsup_n (P^{(n)}(s))^{1/n} \lim_n (P^{(k)}(s'))^{1/n} = P[s] = 1. \end{aligned}$$

Therefore we have

PROPOSITION 1. $P[s]=1 \Rightarrow P[ss']=P[s's]=P[s'ss'']=1$ *for all* $s', s'' \in S$.

S is called *left simple* if for all $s \in S$: $Ss=S$ (this means every element of S can be written in the form $s's$). Proposition 1 implies

PROPOSITION 2. (a) *If S is left simple* (*or right simple, or a group*) *then*

$$P[s] = 1 \quad \textit{for one } s \in S \Leftrightarrow P[s] = 1 \quad \textit{for every } s \in S.$$

(b) *If S has a left unit e* ($es=s$ *for all* s) *then*

$$P[e] = 1 \Leftrightarrow P[s] = 1 \quad \textit{for every } s \in S.$$

Let S be a discrete semigroup with a left unit e and (S, P) an A-pair, further put $P' = \frac{1}{2}(P+\delta_e)$ (δ_e is the probability measure concentrated at e, i.e. $\delta_e(e)=1$, $\delta_e(s)=0$ for $e \neq s \in S$). Then P' is a probability measure on S and Supp P' = Supp $P \cup \{e\}$.

PROPOSITION 3. $P[s]=1 \Rightarrow P'[s]=1$.

PROOF. e is a left unit, therefore $\delta_e * P = P$. So

$$P'^{(2n)}(s) \geqq \frac{1}{2^{2n}} \sum_{k=1}^{2n} \frac{1}{2}\binom{2n}{k} P^{(k)}(s) \geqq \frac{1}{4^n}\frac{1}{2}\binom{2n}{n} P^{(n)}(s)$$

and

$$1 \geqq (P'^{(2n)}(s))^{1/2n} \geqq \frac{1}{2}\left(\frac{1}{2}\right)^{1/2n} \binom{2n}{n}^{1/2n} (P^{(n)}(s))^{1/2n} = a_n (P^{(n)}(s))^{1/2n}.$$

Since $\lim_n a_n = 1$ we get $P'[s]=1$.

Proposition 3 says that if (S, P) is an A-pair then so is (S, P') (and we have $P'(e)>0$).

THEOREM 1. *Let S be a discrete semigroup with a left unit e and (S, P) an A-pair. Then*

(1) $P[e]=1 \Rightarrow \lim_{n\to\infty}(P'^{(n)}(s))^{1/n}=1$ *for every* $s \in S$,

(2) $\lim_{n\to\infty}(P'^{(n)}(s))^{1/n}=1 \Leftrightarrow \lim_{n\to\infty}(P'^{(n+1)}(s)/P'^{(n)}(s))=1$.

PROOF. Similar to the proof of Theorem 1 and Theorem 2 of [3].

3. The class of A-semigroups.

THEOREM 2. *Let S be a finite semigroup and P a probability measure on S such that* Supp P *generates S. Then (S, P) is an A-pair.*

PROOF. Let c be the cardinal number of S. Then for every $n=1, 2, \cdots$ there exists an element s_n in S with $P^{(n)}(s_n)\geqq 1/c$. Because S is finite there is an s_0 in S which appears infinitely often in the sequence $s_1, s_2, \cdots$ and so $P^{(n_k)}(s_0)\geqq 1/c$ for some sequence $n_1<n_2<\cdots$ of natural numbers. Therefore $P[s_0]=1$.

THEOREM 3. *Let S be a discrete semigroup with left cancellation ($ss'=ss'' \Rightarrow s'=s''$) and a left unit e. If S is an A-semigroup then S is left amenable.*

PROOF. By assumption there exists a probability measure P on S and an element $s \in S$ such that (1) Supp P generates S and (2) $P[s]=1$. By Proposition 3 we have $P'[s]=1$ ($P'=\frac{1}{2}(P+\delta_e)$).

Now let $x \in l_2(S)$; then $P' * x \in l_2(S)$ and $\|P' * x\|_2 \leqq \|P'\|_1 \|x\|_2 = \|x\|_2$. So we can consider $P' *$ as an operator on $l_2(S)$ and we have for its norm $\|P' *\|_{2\to 2} \leqq 1$.

Further, $\delta_e \in l_2(S)$. Next,

$$P'(s) = P'(s)\delta_e(e) \leqq \left(\sum_{s\in S}\left(\sum_{s_1 s_2=s} P'(s_1)\delta_e(s_2)\right)^2\right)^{1/2} = \|P' * \delta_e\|_2$$

$$\leqq \sup_{\|x\|_2=1} \|P' * x\|_2 = \|P' *\|_{2\to 2} \leqq 1,$$

and in the same way

$$P'^{(n)}(s) \leqq \|P'^{(n)} *\|_{2\to 2} \leqq 1.$$

So $1=P'[s]\leqq \limsup_n \|P'^{(n)} *\|_{2\to 2}^{1/n} =$ spectral radius of $P' * \leqq \|P' *\|_{2\to 2} \leqq 1$ or $\|P' *\|_{2\to 2}=1$; by the same argument $\|P'^{(k)} *\|_{2\to 2}=1$ for $k=1, 2, \cdots$. But Supp P' generates S and so for every finite $E\subset S$ there exists a natural number k with $E \subset \operatorname{Supp} P'^{(k)}$ and $e \in \operatorname{Supp} P'^{(k)}$. Then [1] (Theorem 1, (e)$\Rightarrow$(a)) implies that S is left amenable.

REMARK 1. For S a group G and P a symmetric probability measure on G whose support generates G we have from the theorem of Kesten and Proposition 2: G amenable $\Rightarrow P[g]=1$ for every $g \in G$. So for groups we lose nothing in considering only symmetric probability measures. If P is not symmetric this implication need no longer be true; consider for example the infinite cyclic group $G=\langle a\rangle$, generated by a. This group is commutative, therefore amenable. Let

$$P = \alpha\delta_a + (1-\alpha)\delta_{a^{-1}} \qquad (0 < \alpha < 1, \alpha \neq \tfrac{1}{2}).$$

Then

$$P^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \delta_{a^{2k-n}} \alpha^k (1-\alpha)^{n-k}$$

and

$$P^{(2n)}(e) = \binom{2n}{n} \alpha^n (1-\alpha)^n.$$

So

$$P[e] = \lim_{n\to\infty} \binom{2n}{n}^{1/2n} (\alpha(1-\alpha))^{1/2} = 2(\alpha(1-\alpha))^{1/2} < 1 \quad \text{for } \alpha \neq \tfrac{1}{2}$$

(and $P[e]=P[g]$ for every $g \in G$ by Proposition 2).

REMARK 2. The statement of Theorem 3 is false for arbitrary semigroups, for there are finite semigroups (which are A-semigroups by Theorem 2) that are not left (or right) amenable.

REMARK 3. The converse of Theorem 3 is not true in general, for there are left amenable semigroups with left cancellation and a unit that are not A-semigroups.

Consider, for example, the infinite cyclic semigroup $S=\{e, a, a^2, \cdots\}$, generated by e (unit) and a. S is abelian and therefore amenable. Let P be a probability measure on S such that Supp P generates S. This implies $0<P(e)=\alpha<1$ and so $P=\alpha\delta_e+(1-\alpha)P_1$, where Supp $P_1 \subset \{a, a^2, \cdots\} = S-\{e\}$.

Then Supp $P_1^{(n)} \subset \{a^n, a^{n+1}, \cdots\}$ and therefore

$$P^{(n)} = \sum_{k=0}^{n} \binom{n}{k} P_1^{(k)} \alpha^{n-k} (1-\alpha)^k.$$

This gives $P^{(n)}(e)=\alpha^n$ and $P[e]=\alpha<1$.

For $l=1, 2, \cdots$ we find for n large enough

$$\begin{aligned} P^{(n)}(a^l) &\leqq \sum_{k=0}^{l} \binom{n}{k} P_1^{(k)}(a^l) \alpha^{n-k} (1-\alpha)^k \\ &\leqq \sum_{k=0}^{l} \binom{n}{k} \alpha^{n-k}(1-\alpha)^k \leqq \alpha^n \binom{n}{l} \sum_{k=0}^{l} \left(\frac{1-\alpha}{\alpha}\right)^k \end{aligned}$$

and therefore $P[a^l] \leqq \alpha < 1$. So S is not an A-semigroup.

THEOREM 4. *The homomorphic image of an A-semigroup is an A-semigroup.*

PROOF. Let (S, P) be an A-pair with $P[s]=1$ $(s \in S)$. Let $\varphi: S \to S_1$ be a homomorphism onto the semigroup S_1. Define the probability measure P_1 on S_1 by

$$P_1(s_1) = P(\varphi^{-1}(s_1)) = \sum_{s \in \varphi^{-1}(s_1)} P(s).$$

Then by induction

$$\begin{aligned} P_1^{(n)}(s_1) &= \sum_{s_1' s_1'' = s_1} \sum_{s' \in \varphi^{-1}(s_1')} P(s') \sum_{s'' \in \varphi^{-1}(s_1'')} P^{(n-1)}(s'') \\ &= \sum_{s's'' \in \varphi^{-1}(s_1)} P(s')P^{(n-1)}(s'') = \sum_{s \in \varphi^{-1}(s_1)} P^{(n)}(s), \end{aligned}$$

and therefore $1 \geqq P_1^{(n)}(s_1) \geqq P^{(n)}(s)$ for $s \in \varphi^{-1}(s_1)$. Thus $P_1[s_1]=1$ if $P[s]=1$ (where $\varphi(s)=s_1$).

THEOREM 5. *Let (S_1, P_1) be an A-pair, (S_2, P_2) be an A-pair such that for some $s_2 \in S_2: \lim_{n\to\infty}(P_2^{(n)}(s_2))^{1/n}=1$. Then $(S_1 \times S_2, P_1 \times P_2)$ is an A-pair.*

PROOF. Supp $P_1 \times P_2$ generates $S_1 \times S_2$ and

$$1 \geqq (P_1 \times P_2)[(s_1, s_2)] \geqq P_1[s_1] \lim (P_2^{(n)}(s_2))^{1/n} = P_1[s_1] = 1$$

for some $s_1 \in S_1$.

EXAMPLE 1. Let S be a countable right zero semigroup ($ss'=s'$ for all $s, s' \in S$). If P is any probability measure on S whose support generates S, then Supp $P=S$ and

$$P^{(n)}(s) = \sum_{s_2=s_1s_2=s} P(s_1)P^{(n-1)}(s_2) = \sum_{s_1 \in S} P(s_1)P^{(n-1)}(s) = P(s).$$

Therefore $P[s]=\lim(P(s))^{1/n}=1$, because $P(s)>0$ for every $s \in S$; so we see that every countable right zero semigroup is an A-semigroup.

EXAMPLE 2. As in Remark 3 one can show that the semigroup $S=\{e, a, b, ab, \cdots\}$, generated by two elements a and b, is not an A-semigroup.

REFERENCES

1. M. M. Day, *Convolutions, means and spectra*, Illinois J. Math. **8** (1964), 100–111. MR **28** #2447.

2. P. Gerl, *Über die Anzahl der Darstellungen von Worten*, Monatsh. Math. **75** (1971), 205–214.

3. ———, *Diskrete, mittelbare Gruppen*, Monatsh. Math. (to appear).

4. H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354. MR **22** #253.
5. ———, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156. MR **22** #2911.

Department of Mathematics, Pahlavi University, Shiraz, Iran

Current address: Mathematisches Institut der Universität, Strudlhofgasse 4, A-1090 Wien, Austria