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# ON CHARACTERISTIC CLASSES OF GROUPS AND BUNDLES OF $K(\Pi, 1)$'S

R. O. HILL, JR.[1]

ABSTRACT. If $F \to E \to B$ is a fibration with $F=K(G,1)$, $G$ Abelian, and $B=K(\Pi, 1)$, then it is shown that the action and characteristic class of the fibration correspond to those of the induced group extension.

1. Let $F$ and $B$ be spaces of the homotopy type of connected CW complexes and which are Eilenberg-Mac Lane spaces of type $K(G, 1)$ and $K(\Pi, 1)$, respectively, where $G$ is Abelian. Given any fibration $F \xrightarrow{i} E \xrightarrow{p} B$, it is natural to ask: what is the fundamental group of $E$? The only nontrivial part of the homotopy sequence for $p$ is

$$0 \longrightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B) \longrightarrow 1.$$

Thus $\pi_1(E)$ is a group extension of $\pi_1(F)$ by $\pi_1(B)$ and $E$ is a $K(\pi_1(E), 1)$). Since $G=\pi_1(F)$ is Abelian, this extension naturally induces a $\pi_1(B)$-module structure on $G$, $\varphi:\pi_1(B)\to \mathrm{Aut}(G)$. See [**10**, p. 108]. The (equivalence class of this) extension is completely determined by a certain 2-dimensional group cohomology characteristic class $c \in H^2_\varphi(\Pi; G)$.

On the other hand, in any fibration, "dragging" the fiber over a loop in the base space induces a homotopy equivalence on the fiber. In this way, the fibration $p$ (geometrically) induces an action $\varphi'$ of $\pi_1(B)$ on $\pi_1(F)$ and the characteristic class of $p$ (i.e. the first obstruction to a cross-section) is a 2-dimensional cohomology class $k \in H^2_{\varphi'}(K(\Pi, 1); \{G\})$ (where this denotes cohomology with local coefficients twisted by $\varphi'$).

If $\varphi=\varphi'$, then $H^2_\varphi(\Pi; G)$ and $H^2_{\varphi'}(K(\Pi, 1); \{G\})$ are naturally isomorphic, and it is natural to conjecture that $c$ corresponds to $k$. Indeed, this is so and in fact we will prove:

THEOREM 1. *Let $F$ and $B$ be as above and $F\to E\to B$ be a fibration.*

(a) *The two actions $\varphi$ and $\varphi'$ of $\pi_1(B)$ on $\pi_1(F)$ are the same.*

(b) *If $c$ and $k$ are as above, then $\Phi(c)=k$, where $\Phi:H^2_\varphi(\Pi; G)\to H^2_\varphi(K(\Pi, 1); \{G\})$ is the natural isomorphism.*

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(c) *This class $k$ is also the characteristic class for the map that pulls the fibration back from the universal example.*

A version of this theorem was proven in the special case of principal bundles (i.e., all actions are trivial) by Massey in [**2**, pp. 37–41]. He showed there was some isomorphism which preserves the characteristic classes, but he left open if it was the canonical one. The above, of course, shows it was. See also Conner and Raymond [**3**, §8] for the essence of another approach.

In §§2 and 3, we set up the algebra and the topology we need. Theorem 1(a) and (b) is proven in §4 and 1(c) is outlined in §5. In §6 we give an application. I would like to thank Frank Nussbaum for our many helpful discussions.

2. In this section we summarize the theory of group extensions. See Mac Lane [**10**, Chapter IV] for details.

If $G$, $H$, and $\Pi$ are groups, an extension of $G$ by $\Pi$ is a short exact sequence $E:1\to G\xrightarrow{i}H\xrightarrow{j}\Pi\to 1$. If $G$ is Abelian, then $E$ naturally induces an action of $\Pi$ on $G$, $\varphi:\Pi\to\operatorname{Aut} G$, by $i(\varphi(k)g)=hgh^{-1}$, where $g\in G$, $h\in H$, $k\in\Pi$ and $j(h)=k$.

Suppose hereafter that $G$ is Abelian, and let $\varphi$ be a fixed action of $\Pi$ on $G$. The set of all equivalence classes of extensions of $G$ by $\Pi$ which induce $\varphi$ forms an Abelian group under Baer sum which is naturally isomorphic to $H^2_\varphi(\Pi; G)$. See [**10**, Chapter IV, 4.1 and 5.2]. We need to know this isomorphism explicitly, so we briefly describe the bar construction.

For $n\geqq 0$, let $B_n$ be the free Abelian group generated by all symbols of the form $x_0[x_1|\cdots|x_n]$, where $x_i\in\Pi$, $x_i\neq 1$ if $i\geqq 1$, called the nonhomogeneous generators. Set $x_0[x_1|\cdots|x_n]=0$ if some $x_i=1$ for $i\geqq 1$. The $Z(\Pi)$ (=group ring) module structure is $z(x_0[x_1|\cdots|x_n])=zx_0[x_1|\cdots|x_n]$. For $n\geqq 1$, define $\partial_n:B_n\to B_{n-1}$ by

$$\partial_n x_0[x_1|\cdots|x_n] = x_0x_1[x_2|\cdots|x_n] + (-1)^n x_0[x_1|\cdots|x_{n-1}] + \sum_{i=1}^{n-1} x_0[x_1|\cdots|x_ix_{i+1}|\cdots|x_n],$$

define $\partial_n^*:\operatorname{Hom}_{Z(\Pi)}(B_{n-1}, G)\to\operatorname{Hom}_{Z(\Pi)}(B_n, G)$, and define $H^n_\varphi(K; G)=\ker\partial^*_{n+1}/\operatorname{im}\partial^*_n$.

In the extension $E:0\to G\xrightarrow{i}H\xrightarrow{j}\Pi\to 1$ (which induces $\varphi$), for each $x\in\Pi$, pick $v(x)\in H$ such that $jv(x)=x$ and $v(1)=1$. Possibly $v(x)v(y)\neq v(xy)$, but for every $x, y\in\Pi$, there is an $f(x, y)\in G$ such that $if(x, y)=v(x)v(y)v(xy)^{-1}$. Define $C_E:B_2\to G$ by $C_E(x_0[x_1|x_2])=\varphi(x_0)\cdot f(x_1, x_2)$.

PROPOSITION 2.1. *The cochain $C_E$ is a cocycle and the mapping defined by $E \to C_E$ is the natural isomorphism between the group of extensions inducing $\varphi$ and $H^2_\varphi(\Pi; G)$.*

PROOF. This is a direct consequence of Mac Lane [**10**, pp. 112–114].

3. Let $W = W(\Pi)$ be the standard free acyclic semisimplicial complex corresponding to the group $\Pi$. Its $n$-simplices are ordered $(n+1)$-tuples of elements of $\Pi$, $[a_0, \cdots, a_n]$. It has face and degeneracy operators

$$\begin{aligned} \partial_i[a_0, \cdots, a_n] &= [a_0, \cdots, a_i a_{i+1}, \cdots, a_n], \qquad 0 \leqq i \leqq n-1, \\ \partial_n[a_0, \cdots, a_n] &= [a_0, \cdots, a_{n-1}], \\ s_i[a_0, \cdots, a_n] &= [a_0, \cdots, a_i, 1, a_{i+1}, \cdots, a_n], \qquad 0 \leqq i \leqq n, \end{aligned}$$

and $\Pi$ acts on $W$ by $a[a_0, \cdots, a_n] = [aa_0, a_1, \cdots, a_n]$. (Note that $[1, a_1, \cdots, a_n]$ with no $a_i = 1$ is not degenerate.)

Let $|W|$ be Milnor's geometric realization [**11**], which is a CW complex with an $n$-cell for each nondegenerate $n$-simplex of $W$. Let $C_* = C_*(|W|)$ be its CW chains. Denote the $n$-cells by $\langle a_0, \cdots, a_n\rangle$ and the corresponding generators of $C_n$ by $(\langle a_0, \cdots, a_n\rangle)$, $a_i \in \Pi$, no $a_i = 1$ for $i > 0$. By the realization and the definitions of $\partial_i$, $\partial: C_n \to C_{n-1}$ is given by

$$\partial(\langle a_0, \cdots, a_n\rangle) = \sum_{i=0}^{n-1} (-1)^i(\langle a_0, \cdots, a_i a_{i+1}, \cdots, a_n\rangle) + (-1)^n(\langle a_0, \cdots, a_{n-1}\rangle). \tag{3.1}$$

The action of $\Pi$ on $W$ induces a free cellular action of $\Pi$ on $|W|$ and a corresponding action on $C_*$. As is well known, $|W|$ is acyclic, $K = |W|/\Pi$ is a $K(\Pi, 1)$, and $q: |W| \to K$, the quotient map, is the universal cover.

Since the action of $\Pi$ on $|W|$ is cellular, $q$ induces a CW structure on $K$. To each equivalence class $\{\langle aa_0, a_1, \cdots, a_n\rangle \mid a \in \Pi\}$ corresponds an $n$-cell of $K$, denoted by $(a_1|\cdots|a_n)$, and a generator of the CW chains of $K$, $C_*(K)$, $((a_1|\cdots|a_n))$, $1 \neq a_i \in \Pi$. Then $q: |W| \to K$ is given by $q\langle a_0, \cdots, a_n\rangle = (a_1|\cdots|a_n)$ and similarly for $q_*: C_*(|W|) \to C_*(K)$. We compute $\partial$ in $C_*(K)$ using $q_*$ and (3.1) and get

$$\partial((a_1|\cdots|a_n)) = ((a_2|\cdots|a_n)) + (-1)^n((a_1|\cdots|a_n)) + \sum_{i=1}^{n-1}((a_1|\cdots|a_i a_{i+1}|\cdots|a_n)).$$

(Compare with Eilenberg and Mac Lane [**6**, §1.3].)

We are now able to give a very complete geometric description of (at least) the 2-skeleton $K^{(2)}$ of $K$.

LEMMA 3.2. (a) *$K$ has one* 0-*cell* (1).

(b) *It has one (oriented)* 1-*cell* $(x)$ *for each* $1 \neq x \in \Pi$, *and of course the attaching maps are trivial.*

(c) *It has one (oriented)* 2-*cell* $(x|y)$ *for each* $x, y \in \Pi$, $x \neq 1 \neq y$. *The attaching map is algebraically given by* $((y))-((xy))+((x))$, *so that the* 2-*cell* $(x|y)$ *is attached with the relation* $(x)+(y)-(xy)=1$.

PROOF. This is immediate from the above construction.

LEMMA 3.3. *We can take the isomorphism* $\Pi \cong \pi_1(K(\Pi, 1), *)$ (*where* $* = (1)$) *as being induced by taking the cell* $(x)$ *to represent* $x \in \Pi$.

This is an easy exercise.

LEMMA 3.4. *Let* $B_* = B_*(\Pi)$ *be the bar resolution described in* §2.

(a) *There is an isomorphism* $B_* \cong C_*(|W|)$.

(b) *This isomorphism induces the natural isomorphism* $\Phi: H^*_\varphi(\Pi; G) \to H^*_\varphi(K(\Pi, 1); \{G\})$ *for the case the* $K(\Pi, 1)$ *is* $K$.

PROOF. (a) Define $B_n \to C_n(|W|)$ by sending the nonhomogeneous generator $x_0[x_1|\cdots|x_n]$ to the generator $(\langle x_0, x_1, \cdots, x_n\rangle)$. The constructions have been set up so that it is trivial to check that this induces an isomorphism of differential graded $Z$-modules.

(b) The usual isomorphism (see [**10**, Chapter IV, 11] or [**3**]) is obtained by observing that the total singular complex of $|K(\Pi, 1)|$, its universal cover, can be interpreted as a free $Z(\Pi)$ (=group ring) resolution of $Z$, so it is naturally chain equivalent to the bar resolution. Consequently the cohomology of the bar resolution, which is $H^*_\varphi(\Pi; G)$, is isomorphic to $H^*_e(|K(\Pi, 1)|; G)$, the equivariant cohomology ,which, in turn, is naturally isomorphic to $H^*_\varphi(K(\Pi, 1); \{G\})$ by Eilenberg [**5**]. For this $K(\Pi, 1)$, $|W(\Pi)|$ is its universal cover, so the result follows easily using the above isomorphism and the usual isomorphism between cellular and singular cohomologies. Indeed, we could even follow this on the chain level using the chain equivalence $\kappa$ defined by Eilenberg and Mac Lane in [**6**, §7].

COROLLARY 3.5. *If* $f: B_2(\Pi) \to G$ *is an (equivariant)* 2-*cocycle, then under the above isomorphisms, the corresponding cellular* 2-*cocycle* $(f) \in C^2(|W(\Pi)|; G)$ *is given by* $(f)(\langle a_0, a_1, a_2\rangle) = f(a_0[a_1|a_2])$ *and the corresponding* 2-*cocycle* $\{f\} \in C^2_\varphi(K; \{G\})$ *is given by* $\{f\}(c_1|c_2) = f(1[c_1|c_2])$.

PROOF. This is immediate from the above.

4. We are now ready to prove Theorem 1.

*Part* (a). This is a corollary of a stronger theorem proven in [**9**]. For the reader's convenience, an independent proof of this fact is included here.

Let $\alpha: I \to B$ represent $[\alpha] \in \pi_1(B, b_0)$ and $A: I \times F \to E$ cover $\alpha^{-1}$. Let $\beta: I \to F$ represent $[\beta] \in \pi_1(F, e_0)$, let $\gamma: (I, 0, 1) \to (F, e_0, e_1)$, where $e_1 = A(1, e_0)$ and let $\zeta: I \to F$ be given by $\zeta(t) = A(1, \beta(t))$. By definition of the geometric action of $\pi_1(B, b_0)$ on $\pi_1(F, e_0)$, $\varphi'$, $\gamma(\zeta)\gamma^{-1}$ represents $(\varphi'[\alpha])[\beta]$.

Let $\sigma = A | I \times e_0$, which is the path in $E$ (from $e_0$ to $e_1$) that the basepoint follows under $A$. Note that $\sigma$ covers $\alpha^{-1}$, so that $(i\gamma)\sigma^{-1}$ is a loop in $E$ based at $e_0$ which covers $w\alpha \sim \alpha$ (where $w$ is the constant loop at $b_0$). By definition of the algebraic action, $\varphi$, $[(i\gamma)\sigma^{-1}]\beta[(i\gamma)\sigma^{-1}]^{-1}$ represents $i_*(\varphi[\alpha][\beta])$. But it is straightforward to use $A(1 \times \beta): I \times I \to E$ to show

$$i(\zeta) \sim \sigma^{-1}\beta\sigma: (I, \dot{I}) \to (E, e_0).$$

Therefore

$$i[\gamma(\zeta)\gamma^{-1}] \sim [(i\gamma)\sigma^{-1}]\beta[\sigma(i\gamma)^{-1}]$$

so we are done.

*Part* (b). Suppose $K(G, 1) \xrightarrow{i} E \xrightarrow{p} K(\Pi, 1)$ is a fibration. Let $G$ denote $\pi_1(K(G, 1), e_0)$, assume $K(\Pi, 1)$ is the $K$ in §3 (the case of an arbitrary $K(\Pi, 1)$ follows from this using homotopy equivalences), and let $\varphi$ be the induced action of $\pi_1(K, b_0)$ on $G$.

We first represent the algebraic characteristic class. For each $x \in \pi_1(K, b_0)$, pick $u(x) \in \pi_1(E, e_0)$ such that $p_*(u(x)) = x$, choosing $u(1) = 1$. Then $p_*[u(x)u(y)u(xy)^{-1}] = 1$, so for every $x, y \in \pi_1(K, b_0)$, there is a $f(x, y) \in G$ such that $i_*f(x, y) = u(x)u(y)u(xy)^{-1}$. By 2.1, $c: B_2(\pi_1(K, b_0)) \to G$ given by $c(z[x|y]) = \varphi(z)f(x, y)$ is a 2-cocycle representing the characteristic class of the extension.

To represent the geometric characteristic class, we must choose specific representatives. By 3.3, each $x \in \pi_1(K, b_0)$ is represented by the cell $(x)$. For each $x \in \pi_1(K, b_0)$, pick a loop $\alpha_x \in u(x) \in \pi_1(E, e_0)$, picking the constant loop for $\alpha_1$. Then $p\alpha_x \sim (x)$. Construct a map $s$ from the 1-skeleton of $K$ to $E$ by letting $s$ on the loop $(x)$ be the loop $\alpha_x$. Thus $s$ is a homotopy cross-section, and we ask: can $s$ be extended to the 2-skeleton? We examine this using classical obstruction theory (see Steenrod [**14**]). By 3.2, $K$ has one 2-cell $(x|y)$ for each $x, y \in \Pi$, $x \neq 1 \neq y$, so oriented that $\partial(x|y) = (x) + (y) - (xy)$ (where this is path composition). Therefore $p_*(\alpha_x + \alpha_y - \alpha_{xy})$ is null-homotopic in $B$, so there is a loop $g(x, y)$ in $K(G, 1)$ such that $i_*g(x, y) = \alpha_x + \alpha_y - \alpha_{xy}$. Since $i_*$ is 1-1, the loop $g(x, y)$ represents $f(x, y)$, by construction. By the definition of the obstruction, therefore, the homomorphism $k: C_2(K) \to G$ given by $((x|y)) \to f(x, y)$ is the cocycle which represents the geometric characteristic class in $H^2_\varphi(K; \{G\})$ of this fibration. But, using 3.5, a simple comparison now shows that $k$ is the image of $c$, above, under the canonical isomorphism.

5. Just as $K(G, n+1)$ and a universal class $v \in H^{n+1}(K(G, n+1); G)$, and the loop-path fibration classify cohomology and principal fibrations,

there is a space $L = L_{\mathrm{Aut}\,G}(G, n+1)$ and a universal class $V \in H^{n+1}_{\varphi}(L; \{G\})$ and a fibration $K(G, n) \to K(\mathrm{Aut}\, G, 1) \xrightarrow{q} L$ which classify twisted cohomology and twisted fibrations. See [8], [12], [13], [9] for details. $L$ is constructed from $K(G, n+1)$; $q$, from the loop-path fibration; and $V$, from $v$. By those constructions, $V$ is the obstruction for a cross-section to $q$, since $v$ is the obstruction for the loop-path fibration. For completeness we observe:

PROPOSITION 5.1. *If $K(G, 1) \to K(E, 1) \to K(\Pi, 1) = K$ is the pull-back of $q$ by a map $f: K(\Pi, 1) \to L_{\mathrm{Aut}\,G}(G, 2)$ such that $\varphi = f_*: \pi_1(K) \to \pi_1(L)$, then $f^*(V) \in H^2_{\varphi}(K(\Pi, 1); \{G\})$ is also the obstruction $k$.*

PROOF. Obstructions are preserved under pull-backs.

6. Let $B$ be a compact, connected 2-manifold which is not the 2-sphere or the real projective plane, so that $B$ is a $K(\pi_1(B, *), 1)$ and its Euler characteristic $\chi_B$ is $\leqq 0$. If $F \to E \to B$ is a fibration with $F$ a $K(G, 1)$ it may then be possible to use Theorem 1 to give a purely algebraic description of $\pi_1(E)$. As illustration, let $S^1 \to E \xrightarrow{p} B$ be the bundle of unit tangent vectors and let $\varphi: \pi_1(B, *) \to \mathrm{Aut}(\pi_1(S^1, *) \simeq Z)$ be the action which is naturally induced by $p$ (which of course is trivial iff $B$ is orientable). Then $H^2_{\varphi}(B, \{Z\}) \simeq Z$ with the fundamental cocycle $\mu$ as generator, and $\chi_B \mu$ is the characteristic class for the bundle $p$ (see Steenrod [**14**, pp. 200, 201] and Aleksandroff and Hopf [**1**, pp. 548–552]).

A straightforward application of Theorem 1 now gives:

THEOREM 6.1. *If $B$ is as above and $S^1 \to E \xrightarrow{p} B$ is the bundle of unit tangent vectors to $B$ and $\varphi$ is the action of $\pi_1(B, *)$ on $\pi_1(S^1)$ induced by $p$, then*

$$0 \longrightarrow Z \longrightarrow \pi_1(E) \xrightarrow{p_*} \pi_1(E) \longrightarrow 1$$

*is the group determined by $\chi_B u \in H^2_{\varphi}(\pi_1(B); Z) \simeq Z$, where $u$ is the image of the fundamental cocycle under the natural isomorphism $\Phi$.*

Up to a factor of $\pm 1$, this was proven by Massey for the orientable case in [**2**, Theorem 3].

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823