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IDENTITIES FOR SERIES OF THE TYPE $\sum f(n)\mu(n)n^{-s}$

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ABSTRACT. Identities are obtained relating the series of the title with $\sum f(n)\mu(n)\mu(p, n)n^{-s}$ where f is completely multiplicative, $|f(n)|\leqq 1$, and p is prime. Applications are given to vanishing subseries of $\sum \mu(n)/n$.

1. **Introduction.** Von Mangoldt [7] and Landau [3] proved that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0, \tag{1}$$

where $\mu(n)$ is the Möbius function. Landau [5] later showed that (1) is equivalent to the prime number theorem. Kluyver [2] described a method for evaluating subseries of (1) of the form

$$\sum_{m=0}^{\infty} \frac{\mu(mb + h)}{mb + h}, \tag{2}$$

where $0<h\leqq b$, although he did not prove convergence of these subseries. Landau [4] proved convergence and expressed the subseries (2) as a linear combination of reciprocals of Dirichlet L-functions.

The results of Landau and Kluyver imply the formulas

$$\sum_{n=1;\ n\equiv 0 \pmod p}^{\infty} \frac{\mu(n)}{n} = 0, \qquad \sum_{n=1;\ n\not\equiv 0 \pmod p}^{\infty} \frac{\mu(n)}{n} = 0, \tag{3}$$

for every prime p. In this note we obtain some identities for Dirichlet series, one of which gives a new proof of (3).

THEOREM 1. *For any prime p and any complex $s=\sigma+it$ with $\sigma\geqq 1$ we have*

$$(1 + p^{-s})\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = (1 - p^{-s})\sum_{n=1}^{\infty} \frac{\mu(n)\mu(p, n)}{n^s}, \tag{4}$$

where $\mu(p, n)$ denotes the Möbius function evaluated at the g.c.d. of p and n.

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Since

$$\mu(p, n) = \begin{cases} 1 & \text{if } p \nmid n, \\ -1 & \text{if } p \mid n, \end{cases}$$

the relations in (3) follow by taking $s=1$ in (4) and using (1).

A special case of Theorem 1 with $p=2$ was recently discovered by Tord Hall [**1**]. Although Hall's method can be adapted to prove Theorem 1, the proof given here seems more natural. It is based on the following property of the Möbius function.

LEMMA 1. *For every prime p we have*

$$\sum_{d \mid n} \mu(d)\mu(p, d) = 1 \quad \text{if } n = 1,$$
$$= 2 \quad \text{if } n = p^a, \quad a \geqq 1,$$
$$= 0 \quad \text{otherwise.}$$

2. **Proof of Lemma 1.** If $n=1$ the proof is immediate. If $n>1$ we have

$$\sum_{d \mid n} \mu(d)\mu(p, d) = \sum_{d \mid n; p \nmid d} \mu(d) - \sum_{d \mid n; p \mid d} \mu(d)$$
$$= \sum_{d \mid n} \mu(d) - 2 \sum_{d \mid n; p \mid d} \mu(d) = -2 \sum_{d \mid n; p \mid d} \mu(d),$$

since $n>1$. If $p \nmid n$ the last sum is empty and hence equals zero. If $p \mid n$ then $n=p^a q$ where $a \geqq 1$, $(q, p)=1$, $q>1$. Every divisior of n divisible by p has the form $p^t\delta$ where $1 \leqq t \leqq a$ and $\delta \mid q$. Hence the last sum is

$$-2 \sum_{t=1}^{a} \sum_{\delta \mid q} \mu(p^t\delta) = -2 \sum_{\delta \mid q} \mu(p\delta) = 2 \sum_{\delta \mid q} \mu(\delta) = 2 \quad \text{if } q = 1,$$
$$= 0 \quad \text{if } q > 1.$$

This proves Lemma 1.

3. **Proof of Theorem 1.** The sum in Lemma 1 is the coefficient of n^{-s} in the Dirichlet series obtained by multiplying $\sum \mu(n)\mu(p, n)n^{-s}$ by $\zeta(s) = \sum n^{-s}$. Therefore if $\sigma>1$ we have

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)\mu(p, n)}{n^s} = 1 + 2 \sum_{a=1}^{\infty} \frac{1}{p^{as}}. \tag{5}$$

The Dirichlet series on the right of (5) is also a geometric series which converges absolutely for $\sigma>0$ and has sum

$$1 + 2p^{-s}/(1 - p^{-s}) = (1 + p^{-s})/(1 - p^{-s}).$$

Since $1/\zeta(s)=\sum \mu(n)n^{-s}$, equation (5) is equivalent to (4) for $\sigma>1$. Now the series $\sum \mu(n)\mu(p, n)n^{-s}$ also converges for $\sigma=1$ since it is the product of the Dirichlet series $\sum \mu(n)n^{-s}$, convergent for $\sigma\geqq 1$, and a Dirichlet series which converges absolutely for $\sigma>0$. (See Landau [6, §185].) Therefore the identity in (4) is valid for $\sigma\geqq 1$.

Theorem 1 can be extended as follows.

THEOREM 2. *Let f be a completely multiplicative function with $|f(n)|\leqq 1$ for all $n\geqq 1$. Then for any prime p and any complex $s=\sigma+it$ with $\sigma>1$ we have*

$$(6)\qquad (1+f(p)p^{-s})\sum_{n=1}^{\infty}\frac{f(n)\mu(n)}{n^s}=(1-f(p)p^{-s})\sum_{n=1}^{\infty}\frac{f(n)\mu(n)\mu(p, n)}{n^s}.$$

Moreover, if the series $\sum f(n)\mu(n)n^{-s}$ converges for $\sigma\geqq c$ for some c with $0<c\leqq 1$, then (6) also holds for $\sigma\geqq c$.

PROOF. If $f(n)=0$ for all n the result holds trivially. If not, then $f(1)=1$ and by Lemma 1 we have

$$\begin{aligned}\sum_{d|n} f(d)\mu(d)\mu(p, d)f(n/d)=f(n)\sum_{d|n}\mu(d)\mu(p, d) &= 1 \qquad \text{if } n=1,\\ &= 2f(p)^a \\ &\qquad\qquad \text{if } n=p^a,\quad a\geqq 1,\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

This identity implies, for $\sigma>1$,

$$\left(\sum_{n=1}^{\infty}\frac{f(n)\mu(n)\mu(p, n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\right)=1+2\sum_{a=1}^{\infty}\frac{f(p)^a}{p^{as}}.$$

Since $|f(n)|\leqq 1$, each Dirichlet series on the left converges absolutely for $\sigma>1$, and the geometric series on the right converges absolutely for $\sigma>0$ to the sum $(1+f(p)p^{-s})/(1-f(p)p^{-s})$. Also, $\sum_{n=1}^{\infty} f(n)n^{-s}\neq 0$ for $\sigma>1$ since it has an Euler product, and

$$\left(\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\right)^{-1}=\sum_{n=1}^{\infty}\frac{\mu(n)f(n)}{n^s}.$$

The rest of the proof is like that of Theorem 1.

4. **Related results.** Sats 2 in Tord Hall's paper is the special case $p=2$ of the following identity.

THEOREM 3. *For any prime p and $s=\sigma+it$ with $\sigma>1$, we have*

$$(1-p^{-s})\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s}=(1+p^{-s})\sum_{n=1}^{\infty}\frac{|\mu(n)|\,\mu(p, n)}{n^s}.$$

This theorem can be proved by Hall's method or by use of the following arithmetical identity.

LEMMA 2. *For all $n \geqq 1$ and any prime p we have*

$$\sum_{d|n} a(d)\,|\mu(n/d)| = \sum_{d|n} b(d)\,|\mu(n/d)|\,\mu(p, n/d), \tag{7}$$

where

$$\begin{aligned} a(n) &= 1 \quad \text{if } n = 1, & b(n) &= 1 \quad \text{if } n = 1,\\ &= -1 \quad \text{if } n = p, & &= 1 \quad \text{if } n = p,\\ &= 0 \quad \text{otherwise}, & &= 0 \quad \text{otherwise}. \end{aligned}$$

It is clear that Lemma 2 implies Theorem 3. To prove Lemma 2 we need only consider three cases: $n=1$; $n=pq$ with $(p,q)=1$; and $n=p^2q$. In all other cases each sum in (7) contains only the term $|\mu(n)|$, corresponding to $d=1$, the other terms being zero. If $n=1$ the result is trivial. If $n=pq$ with $(p,q)=1$, it is easily verified that each sum in (7) is zero. In the remaining case, $n=p^2q$, each sum is equal to $-|\mu(pq)|$.

By a similar argument, Lemma 2 implies the following extension of Theorem 3.

THEOREM 4. *Let f be completely multiplicative with $|f(n)| \leqq 1$ for all n. Then for any prime p and any complex $s=\sigma+it$ with $\sigma>1$ we have*

$$(1 - f(p)p^{-s}) \sum_{n=1}^{\infty} \frac{f(n)\,|\mu(n)|}{n^s} = (1 + f(p)p^{-s}) \sum_{n=1}^{\infty} \frac{f(n)\,|\mu(n)|\,\mu(p, n)}{n^s}.$$

Note. By differentiating (4) for $\sigma>1$, letting $s\to 1+$, and using the relation ([**6**, §159])

$$\sum_{n=1}^{\infty} \frac{\mu(n)\log n}{n} = -1, \tag{8}$$

we find that

$$\sum_{n=1}^{\infty} \frac{\mu(n)\mu(p, n)\log n}{n} = \frac{p+1}{p-1}$$

for every prime p. This implies that each of the following subseries of (8) converges to the sum indicated:

$$\sum_{n=1;\,n\equiv 0 \pmod p}^{\infty} \frac{\mu(n)\log n}{n} = \frac{1}{p-1}, \qquad \sum_{n=1;\,n\not\equiv 0 \pmod p}^{\infty} \frac{\mu(n)\log n}{n} = \frac{p}{1-p}.$$

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