

PERMANENT GROUPS. II

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ABSTRACT. A permanent group is a group of nonsingular matrices on which the permanent function is multiplicative. We consider only permanent groups which contain the group of nonsingular diagonal matrices. If the underlying field is infinite of characteristic zero or greater than n , then each such permanent group consists only of matrices in which exactly one diagonal has all nonzero entries.

A permanent group is a group of nonsingular matrices on which the permanent function is multiplicative. One example is $\mathcal{D}_n(F)$, the group of nonsingular $n \times n$ diagonal matrices over the field F . Marcus and Minc [3] conjectured that Δ_n , the groups of $n \times n$ matrices of the form PD where P is a permutation matrix and $D \in \mathcal{D}_n(F)$ is a maximal permanent group. In this conjecture the field F was not specified and the first author [1] verified the conjecture for the field of complex numbers. In this paper we consider the set $\mathcal{P}_n(F)$ of those permanent groups which contain $\mathcal{D}_n(F)$, and characterize $\mathcal{P}_n(F)$ when F is an infinite field with $\text{char } F = 0$ or $> n$.

In [2] we defined the set $\mathcal{A}_n(F)$ of all $n \times n$ matrix groups G of nonsingular matrices over F satisfying:

- (i) $\mathcal{D}_n(F) \leq G$, and
- (ii) $A, B \in G$ implies $A \circ B^T \in \mathcal{D}_n(F)$

where " \circ " denotes the Hadamard product. Here, we are also concerned with the set $\mathcal{C}_n(F)$ of $n \times n$ nonsingular matrix groups over F such that $G = H \cdot K = \{hk, h \in H, k \in K\}$ where:

- (i) $H \in \mathcal{A}_n(F)$,
- (ii) K is a group of $n \times n$ permutation matrices, and
- (iii) $\langle PHP^{-1} \mid P \in K \rangle \in \mathcal{A}_n(F)$.

Note that $\mathcal{D}_n(F) \leq H$ for every $H \in \mathcal{A}_n(F)$ so that $\mathcal{D}_n(F) \leq G$ for every $G \in \mathcal{C}_n(F)$.

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Yet another set of matrix groups is $\mathcal{B}_n(F)$, consisting of all groups of $n \times n$ nonsingular matrices over F such that:

- (i) $\mathcal{D}_n(F) \leq G$, and
- (ii) $A \in G$ implies A has a unique nonzero diagonal where, by an abuse of language, *nonzero diagonal* shall mean every entry of the diagonal is nonzero.

If A is any $n \times n$ matrix with at least one nonzero diagonal we define $P_A(Q_A)$ to be any one of the permutation matrices such that $P_A A (A Q_A)$ has nonzero main diagonal. The number of nonzero entries in row i (column j) of a matrix B will be denoted by $r_i(B)$ ($c_j(B)$). The (i, j) entry of the product of several matrices A, B, C will be denoted $(ABC)_{ij}$. The set $\{1, \dots, n\}$ will be denoted by $[n]$ and throughout $n > 1$.

THEOREM. *If F is an infinite field and $\text{char } F = 0$ or $> n$ then $\mathcal{P}_n(F) = \mathcal{B}_n(F) = \mathcal{C}_n(F)$.*

We demonstrate the inclusions $\mathcal{C}_n(F) \subseteq \mathcal{P}_n(F)$, $\mathcal{P}_n(F) \subseteq \mathcal{B}_n(F)$, and $\mathcal{B}_n(F) \subseteq \mathcal{C}_n(F)$ in Propositions 1, 3, and 6 respectively with varying restrictions on F . The hypothesis of our theorem implies the hypotheses in each of these propositions.

1. PROPOSITION. *If F is a field with at least 3 elements then $\mathcal{C}_n(F) \subseteq \mathcal{P}_n(F)$.*

PROOF. If $A, B \in G = H \cdot K \in \mathcal{C}_n(F)$ let

$$A = A_1 P_1 A_2 P_2 \cdots A_s P_s \quad \text{and} \quad B = B_1 Q_1 B_2 Q_2 \cdots B_t Q_t$$

where $A_i \in H$, $P_i \in K$ for $i \in [s]$ and $B_j \in H$, $Q_j \in K$ for $j \in [t]$. Define the products

$$\begin{aligned} \hat{P}_i &= P_i P_{i+1} \cdots P_s, & i \in [s], \\ \hat{Q}_j &= Q_1 Q_2 \cdots Q_{j-1}, & j \in [t] \setminus \{1\}, \\ \hat{Q}_1 &= I_n. \end{aligned}$$

Then

$$\begin{aligned} A &= A_1 \hat{P}_1 \hat{P}_2^{-1} A_2 \hat{P}_2 \hat{P}_3^{-1} A_3 \cdots A_{s-1} \hat{P}_{s-1} \hat{P}_s^{-1} A_s \hat{P}_s, \\ B &= \hat{Q}_1 B_1 \hat{Q}_1^{-1} \hat{Q}_2 B_2 \hat{Q}_2^{-1} \cdots B_{t-1} \hat{Q}_{t-1}^{-1} \hat{Q}_t B_t Q_t. \end{aligned}$$

It is immediate that $\hat{P}_1^{-1} A \in \langle PHP^{-1} \mid P \in K \rangle$ and $B \hat{Q}_t^{-1} \hat{Q}_t^{-1} \in \langle PHP^{-1} \mid P \in K \rangle$ while $\langle PHP^{-1} \mid P \in K \rangle \in \mathcal{A}_n(F)$.

Since F has at least 3 elements, $\mathcal{A}_n(F) \subseteq \mathcal{P}_n(F)$ [2, Theorem 3.1]. Therefore,

$$\begin{aligned} \text{per } AB &= \text{per}(\hat{P}_1^{-1} A B \hat{Q}_t^{-1} \hat{Q}_t^{-1}) \\ &= \text{per}(\hat{P}_1^{-1} A) \text{per}(B \hat{Q}_t^{-1} \hat{Q}_t^{-1}) = \text{per } A \text{ per } B. \end{aligned}$$

2. LEMMA. Let F be an infinite field and $A=(a_{ij})$, $B=(b_{ij})$ be $n \times n$ matrices over F . If $J \subseteq [n] \times [n]$ and $(i, j) \in J$ always implies there exists $k \in [n]$ such that $a_{ik}b_{kj} \neq 0$ then there exists $D \in \mathcal{D}_n(F)$ for which $(ADB)_{ij} \neq 0$ for each $(i, j) \in J$.

PROOF. Let $D = \text{diag}[x_1, \dots, x_n]$ be a diagonal matrix of indeterminates x_i , $i \in [n]$, over F and consider the set of polynomials

$$(2.1) \quad f_{ij}(x_1, \dots, x_n) = (ADB)_{ij} = \sum_{s=1}^n a_{is}x_sb_{sj} \quad (i, j) \in J.$$

By hypothesis there is $k \in [n]$ such that $a_{ik}b_{kj} \neq 0$ so that each f_{ij} is nonzero. Hence $f = \prod_{(i,j) \in J} f_{ij}$ is a nonzero polynomial. Since F is infinite there is an n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of elements from F such that $f(\alpha) \neq 0$ [4, Theorem 14, p. 38]. If, say, $\alpha_i = 0$ let

$$g(x) = f(\alpha_1, \dots, \alpha_{i-1}, x, \alpha_{i+1}, \dots, \alpha_n).$$

Because F is infinite there is a nonzero β in F such that $g(\beta) \neq 0$. Accordingly we may assume $\alpha_i \neq 0$ for all $i \in [n]$ and choose $D = \text{diag}[\alpha_1, \dots, \alpha_n]$.

3. PROPOSITION. If F is an infinite field and $\text{char } F = 0$ or $> n$ then $\mathcal{P}_n(F) \subseteq \mathcal{B}_n(F)$.

PROOF. Suppose a group $G \in \mathcal{P}_n(F)$ contains a matrix A with at least 2 distinct nonzero diagonals determined by the permutations σ and τ . Consider an $i \in [n]$ such that $\sigma(i) \neq \tau(i)$ and $C \in G$ such that $r_i(C) = \max\{r_i(B) \mid B \in G\}$. Since $A \in G$ and $r_i(A) \geq 2$, $r_i(C) \geq 2$. If $i \neq 1$ choose a permutation matrix P interchanging 1 and i . Then $PGP^{-1} \in \mathcal{P}_n(F)$, $r_1(PCP^{-1}) \geq r_1(PBP^{-1})$ for all $B \in G$, and $r_1(PCP^{-1}) \geq 2$ since $r_1(PAP^{-1}) = r_i(A) \geq 2$. Thus, without loss of generality we may take $i = 1$.

Let $C = (c_{ij})$, $C^{-1} = (c'_{ij})$, and π be a permutation which corresponds to a nonzero diagonal of C^{-1} ; i.e., $c'_{i\pi^{-1}(i)} \neq 0$ for all $i \in [n]$. In Lemma 2 take $A = C$, $B = C^{-1}$, and

$$J = \{(i, i) \mid i \in [n]\} \cup \{(1, \pi^{-1}(t)) \mid c_{1t} \neq 0\}.$$

Lemma 2 then yields $D \in \mathcal{D}_n(F)$ such that $E = CDC^{-1}$ has nonzero main diagonal and $r_1(E) = r_1(C)$, since for all $i \in [n]$ there exists $k \in [n]$ such that $c_{ik}c'_{ki} \neq 0$ (because $\sum_{k=1}^n c_{ik}c'_{ki} = 1$) and since for t such that $c_{1t} \neq 0$ we have $c_{1t}c'_{t\pi^{-1}(t)} \neq 0$.

Let $E = (e_{ij})$. In particular $e_{11} \neq 0$ so we may assume $e_{1j} \neq 0$ for $j \in [m]$ and $e_{1j} = 0$ for $j \in [n] \setminus [m]$ since, if not, there exists a permutation matrix P fixing row 1 and column 1 in PEP^{-1} and which gives PEP^{-1} the desired form. This is possible because PGP^{-1} is also in $\mathcal{P}_n(F)$.

For arbitrary $B \in G$, $BQ_B = (b'_{ij})$ has the form

$$(3.1) \quad \begin{pmatrix} B_1 & 0 \\ B_3 & B_4 \end{pmatrix}$$

where B_1 is $m \times m$ and the zero block may be empty; otherwise, if $b'_{st} \neq 0$ for some $s \in [m]$ and $t \in [n] \setminus [m]$ then Lemma 2 implies the existence of $D \in \mathcal{D}_n(D)$ such that $EDBQ_B$ has more than m nonzero entries in row 1, contradicting $r_1(E) \geq r_1(B)$ for all $B \in G$. This follows upon taking $A = E$, $B = BQ_B = (b'_{ij})$, and $J = \{(1, i) | i \in [m] \cup \{t\}\}$ in Lemma 2. For, $e_{1i} \cdot b'_{ii} \neq 0$ when $i \in [m]$ and BQ_B has nonzero main diagonal while $e_{is}b'_{st} \neq 0$ since $s \in [m]$.

Now A has 2 nonzero diagonals σ and τ such that $\sigma(1) \neq \tau(1)$. There exists $i \neq 1$ such that $\sigma(1) = \tau(i)$; i.e., $a_{1\sigma(1)} \neq 0$ and $a_{i\sigma(1)} = a_{ir(i)} \neq 0$. Choosing Q_A so that $(AQ_A)_{11} = a_{1\sigma(1)}$ and $(AQ_A)_{i1} = a_{i\sigma(1)}$ we find that $\max\{c_1(BQ_B) | B \in G\} \geq 2$ and so $r = \max\{c_1(B_1) | B \in G\} \geq 2$ in $BQ_B = \begin{pmatrix} B_1 & 0 \\ B_3 & B_4 \end{pmatrix}$.

Let $T \in G$ satisfy $c_1(T_1) = \max\{c_1(B_1) | B \in G\}$ where T_1 is the upper left-hand block in the decomposition (3.1) of T . By Lemma 2 there exists $D \in \mathcal{D}_n(F)$ such that $(TDT^{-1})_{ij} \neq 0$ when $i = j \in [n]$ and $c_1(TDT^{-1}) = c_1(T_1)$. Let $F = TDT^{-1}$. In addition we may assume that $f_{j1} \neq 0$ for $j \in [r]$ and $f_{j1} = 0$ for $j \in [m] \setminus [r]$ since, if not, there exists a permutation matrix P which fixes rows and columns 1, $m+1, \dots, n$ and gives $PF P^{-1}$ the desired form. This is possible because PGP^{-1} is also in $\mathcal{P}_n(F)$; and PEP^{-1} has the same form as E since P does not permute columns $[n] \setminus [m]$.

An argument similar to the one showing that for arbitrary $B \in G$ the matrix BQ_B has the form (3.1) shows that $P_B B$ has the further decomposition

$$(3.2) \quad \left(\begin{array}{cc|c} B_{11} & B_{12} & B_2 \\ 0 & B_{13} & \\ \hline B_3 & & B_4 \end{array} \right)$$

where the zero block may be empty.

Another application of Lemma 2 yields $D \in \mathcal{D}_n(F)$ such that $K = FDE$ has nonzero main diagonal and the form

$$(3.3) \quad \left(\begin{array}{cc|c} K_{11} & K_{12} & 0 \\ 0 & K_{13} & \\ \hline & K_2 & K_3 \end{array} \right)$$

where K_{11} is an $r \times r$ block with no zero entries and $r \geq 2$. The existence of the zero blocks follows immediately from (3.1) and (3.2) since we can take $P_K = Q_K = I$.

Letting $A=K_{11}$ and $B=K_{11}^{-1}$, Lemma 2 ensures the existence of $D_{11} \in \mathcal{D}(F)$ such that $K_{11}D_{11}K_{11}^{-1}$ has all entries nonzero since K_{11} does. Letting $D = \begin{pmatrix} D_{11} & 0 \\ 0 & I \end{pmatrix}$ we have that

$$\mathcal{U} = KDK^{-1} \left(\begin{array}{cc|c} K_{11}D_{11}K_{11}^{-1} & \mathcal{U}_{12} & 0 \\ 0 & I & \\ \hline & \mathcal{U}_2 & I \end{array} \right)$$

and hence for a diagonal matrix $X = \begin{pmatrix} X_{11} & 0 \\ 0 & I \end{pmatrix}$ it is apparent that $\text{per } \mathcal{U}X\mathcal{U} = (\text{per } \mathcal{U})^2 \text{ per } X$ if and only if $\text{per}(\mathcal{U}_{11}X_{11}\mathcal{U}_{11}) = (\text{per } \mathcal{U}_{11})^2 \text{ per } X_{11}$. If $X_{11} = \text{diag}[x, 1, \dots, 1]$ and x is an indeterminate over F then the coefficient of x^r in the polynomial $k(x) = \text{per } \mathcal{U}_{11}X_{11}\mathcal{U}_{11} - (\text{per } \mathcal{U}_{11})^2 x$ is $r! \prod_{i=1}^r U_{i1}U_{1\sigma(i)}$ which is nonzero for $\text{char } F = 0$ or $\text{char } F > n \geq r$. Since F is an infinite field there is a nonzero $\alpha \in F$ such that $k(\alpha) \neq 0$. This contradicts the supposition that some $G \in \mathcal{P}_n(F)$ contains a matrix with more than one nonzero diagonal.

4. LEMMA. Let $A=(a_{ij})$, $B_1=(b_{ij})$, $B'_1=(b'_{ij})$ and $B_2=(c_{ij})$ be $n \times n$ matrices over an infinite field F satisfying:

- (i) A has at least one nonzero diagonal, and
- (ii) $b_{ij} \neq 0$ implies $b'_{ij} \neq 0$, $i, j \in [n]$.

There exist $D_1, D_2 \in \mathcal{D}_n(F)$ such that $(B_1Q_A^{-1}B_2)_{ij} \neq 0$ implies

$$(B'_1D_1AD_2B_2)_{ij} \neq 0, \quad i, j \in [n].$$

PROOF. Let $D_1 = \text{diag}[x_1, \dots, x_n]$ and $D_2 = \text{diag}[y_1, \dots, y_n]$ where the diagonal entries are indeterminates over F . Now,

$$(B_1Q_A^{-1}B_2)_{ij} = \sum_{k=1}^n b_{ik}c_{\sigma^{-1}(k)j} \quad \text{and} \quad (B'_1D_1AD_2B_2)_{ij} = \sum_{k=1}^n \sum_{t=1}^n b'_{ik}x_k a_{kt} y_t c_{tj}.$$

Consider the set $\mathcal{M} = \{(B'_1D_1AD_2B_2)_{ij} \mid (B_1Q_A^{-1}B_2)_{ij} \neq 0\}$ of polynomials in the variables $x_1, \dots, x_n, y_1, \dots, y_n$. Each polynomial in \mathcal{M} has the form

$$b'_{is}x_s a_{s\sigma^{-1}(s)} y_{\sigma^{-1}(s)} c_{\sigma^{-1}(s)j} + f_{ij}(x_1, \dots, x_n, y_1, \dots, y_n)$$

where the polynomial f_{ij} has no terms in $x_s y_{\sigma^{-1}(s)}$ and s may be chosen so that $b'_{is} a_{s\sigma^{-1}(s)} c_{\sigma^{-1}(s)j} \neq 0$. This is possible because $(B_1Q_A^{-1}B_2)_{ij} = \sum_{k=1}^n b_{ik} c_{\sigma^{-1}(k)j} \neq 0$ so for some s , $b_{is} c_{\sigma^{-1}(s)j} \neq 0$ and by (ii) we have $b'_{is} c_{\sigma^{-1}(s)j} \neq 0$ and also $0 \neq (AQ_A)_{ss} = a_{s\sigma^{-1}(s)}$, for $s \in [n]$.

Now, $\prod_{f \in \mathcal{M}} f \neq 0$ since each $f \in \mathcal{M}$ is a nonzero polynomial. Hence by [4, Theorem 14, p. 38] there exists a $2n$ -tuple $\alpha = (\alpha_1, \dots, \alpha_{2n})$ of elements in F such that $\prod_{f \in \mathcal{M}} f(\alpha) \neq 0$ because F is infinite and were some $\alpha_i = 0$ we could find $\beta \neq 0$ in F such that

$$\prod_{f \in \mathcal{M}} f(\alpha_1, \dots, \alpha_{i-1}, \beta, \alpha_{i+1}, \dots, \alpha_{2n}) \neq 0.$$

Accordingly we may assume $\alpha_i \neq 0$ for all $i \in [2n]$ and setting $D_1 = \text{diag}[\alpha_1, \dots, \alpha_n]$ and $D_2 = \text{diag}[\alpha_{n+1}, \dots, \alpha_{2n}]$ we obtain the conclusion.

5. LEMMA. *If G is a maximal group in $\mathcal{B}_n(F)$ and F is an infinite field then $Q_A \in G$.*

PROOF. Suppose G contains A with $Q_A \notin G$. Obviously $Q_A^{-1} \notin G$. The maximality of G implies that $\langle G, Q_A^{-1} \rangle$ contains a matrix with at least two nonzero diagonals, say

$$(5.1) \quad B_1 Q_A^{\alpha_1} B_2 Q_A^{\alpha_2} \cdots B_{m-1} Q_A^{\alpha_{m-1}} B_m$$

where we may take each $\alpha_i = -1$, $i \in [m-1]$.

By Lemma 4 there are $D_1, D_2 \in \mathcal{D}_n(F)$ such that $B_1 D_1 A D_2 B_2$ has nonzero entries at least where $B_1 Q_A^{-1} B_2$ does. Replacing B_1 by $B_1 Q_A^{-1} B_2$ and B'_1 by $B_1 D_1 A D_2 B_2$ in Lemma 4 we see that $(B_1 D_1 A D_2 B_2) \cdot D_3 A D_4 B_3$ has nonzero entries at least where $B_1 Q_A^{-1} B_2 Q_A^{-1} B_3$ does for some $D_3, D_4 \in \mathcal{D}_n(F)$. Repeating this argument as necessary we obtain an element of G , namely $B_1 D_1 A D_2 B_2 \cdots D_{2m-3} A D_{2m-2} B_m$ which will have nonzero entries at least where (5.1) does and hence will have two distinct nonzero diagonals since (5.1) does.

6. PROPOSITION. *If F is an infinite field then $\mathcal{B}_n(F) \subseteq \mathcal{C}_n(F)$.*

PROOF. Let G be a maximal group in $\mathcal{B}_n(F)$. For very $A \in G$ there is exactly one permutation matrix Q_A such that AQ_A has nonzero main diagonal. Let $H = \langle AQ_A | A \in G \rangle$ and $K = \langle Q_A | A \in G \rangle$. By the previous lemma both H and K are subgroups of G and therefore the complex $H \cdot K$ is contained in G . On the other hand each $A \in G$ may be written $A = AQ_A Q_A^{-1}$ which implies $G = H \cdot K$.

If $H \notin \mathcal{A}_n(F)$ there are $A, B \in H$ such that $a_{ij} b_{ji} \neq 0$ and $i \neq j$. Let $D_j = \text{diag}[1, \dots, x, \dots, 1]$ where x is diagonal entry j and set $E = A D_j B$. Since F has at least 3 nonzero elements, x may be chosen so that if $E = (e_{ij})$ then

$$e_{ij} = a_{ij} x b_{jj} + \sum_{k \neq j} a_{ik} b_{kj} \neq 0 \quad \text{and} \quad e_{ji} = a_{jj} x b_{ji} + \sum_{k \neq j} a_{jk} b_{ki} \neq 0.$$

Since H contains $\mathcal{D}_n(F)$ the matrix E is in H so has nonzero main diagonal entries. Therefore E has at least 2 nonzero diagonal entries, contradicting $G \in \mathcal{B}_n(F)$. Hence $H \in \mathcal{A}_n(F)$.

If $A \in H$ then $A = BQ_B$ for some $B \in G$ and has a nonzero main diagonal. Thus any PAP^{-1} with $P \in K$ has a nonzero main diagonal; i.e., $Q_{P^{-1}AP^{-1}} = I$ so that $PAP^{-1} = PAP^{-1}Q_{P^{-1}AP^{-1}} \in H$. Thus $\langle PHP^{-1} | P \in K \rangle \in \mathcal{A}_n(F)$, completing the requirements for $G \in \mathcal{C}_n(F)$.

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