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SOLVABLE AND NILPOTENT NEAR-RING MODULES[1]

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ABSTRACT. The center of a unital near-ring module ${}_RM$ is defined, leading to the construction of a lower central series and a definition of R-nilpotence. Likewise a suitable definition of commutators yields a derived series and R-solvability. When $(R, +)$ is generated by elements which distribute over M the R-nilpotence (R-solvability) is shown to coincide with the nilpotence (solvability) of the underlying group. In this case, nilpotence has implications for R-normalizers and the Frattini submodule.

1. **Introduction.** For basic definitions see [2] or [3]. In this paper, by "near-ring" is meant a right unital near-ring R satisfying $x \cdot 0=0$ for all $x \in R$. Similarly a (left) near-ring module ${}_RM$ over R will always be assumed to be unital. In general a subgroup A of $(M, +)$ is called an R-subgroup if $RA \subseteq A$, and A is an R-submodule if it is a normal R-subgroup satisfying

(SM) For all $r \in R$, $m \in M$, $a \in A$, $r(m+a)-rm \in A$.

In the unital case, a subgroup with property (SM) is in fact a normal subgroup.

ISOMORPHISM THEOREM [2]. *Let $f: M \rightarrow M'$ be an R-epimorphism.*

(i) *If A is an R-subgroup (R-submodule) of M, then $f(A)$ is an R-subgroup (R-submodule) of M'.*

(ii) *If A' is an R-subgroup (R-submodule) of M' then $f^{-1}(A)$ is an R-subgroup (R-submodule) of M.*

(iii) *If A is an R-subgroup (R-submodule) of M containing* $\ker f$, $f^{-1}(f(A))=A$.

A normal series for M is a finite series $M \supset M_1 \supset \cdots \supset M_n=0$ where each M_i is an R-submodule of M_{i-1}. Any two normal series for M have equivalent refinements. M is called simple if it has no proper R-submodules and irreducible if it has no proper R-subgroups. A composition series is a normal series without repetition whose factors are all simple. The Jordan-Hölder theorem holds.

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2. **Nilpotence.** Define the center of ${}_RM$ to be $Z(M)=Z_R(M)=\{a \in M \mid r(b+sa)=rsa+rb$ for all $b \in M$, r, $s \in R\}$. Taking $r=s=1$ we see $Z_R(M)\subseteq Z_Z(M)$. For $x, y \in M$ and $r, s \in R$ define $[y, x, r, s]=r(y+sx)-ry-rsx$ and define the upper central series inductively by $Z_0=0$, $Z_1=Z(M)$, $Z_i=\{x \in M \mid [y, x, r, s] \in Z_{i-1}$ for all $y \in M$, $r, s \in R\}$. Clearly these sets satisfy $Z_i \subseteq Z_{i+1}$ for all i.

THEOREM 2.1. *Z_i is an R-submodule of M for all i and $Z_i/Z_{i-1}=Z(M/Z_{i-1})$.*

PROOF. Inductively, assume Z_{i-1} is an R-submodule. We first show that Z_i is a subgroup of M. If $x, x' \in Z_i$ then there exist $z_j \in Z_{i-1}$ for $j=1, 2, \cdots, 7$ such that

$$\begin{aligned} r(y+s(x-x')) &= r(y+z_1+s(-1)x'+sx) \\ &= z_2+rsx+r(y+z_1+s(-1)x') \\ &= z_2+rsx+z_3+rs(-1)x'+r(y+z_1) \\ &= z_2+rsx+z_3+rs(-1)x'+z_4+ry \\ &= z_5+rsx+rs(-1)x'+ry \\ &= z_5+[z_6+rs(-1)x'+rsx]+ry \\ &= z_5+z_6+z_7+rs(x-x')+ry. \end{aligned}$$

Hence $r(y+s(x-x'))-ry-rs(x-x') \in Z_{i-1}$ as required. Moreover Z_i is clearly an R-subgroup. Also Z_1 has property (SM) since, if $a \in Z(M)$, $r \in R$ and $b \in M$, then $r(b+a)-rb=ra+rb-rb=ra \in Z(M)$. Now suppose inductively that $Z_{i-1}/Z_{i-2}=Z(M/Z_{i-2})$. Then $x \in Z_i$ iff $[y, x, r, s] \in Z_{i-1}$ iff $[y+Z_{i-1}, x+Z_{i-1}, r, s]=Z_{i-1}$ for all $y+Z_{i-1} \in M/Z_{i-1}$. Hence $Z_i/Z_{i-1}=Z(M/Z_{i-1})$ as claimed, and this shows also that Z_i/Z_{i-1} is an R-submodule of M/Z_{i-1}. By the isomorphism theorem therefore, Z_i is an R-submodule of M.

REMARK. An easy calculation shows that $Z(M)$ is the categorical center as defined in [**1**].

Define M to be R-nilpotent of class n if n is the least integer such that $Z_n=M$.

Writing $[y, x, r, 1]=[y, x, r]$ we define the R-commutator of two R-subgroups A and B to be the R-subgroup $[A, B]_R$ (or $[A, B]$) generated by $\{[a, b, r] \mid a \in A, b \in B, r \in R\}$.

PROPOSITION 2.2. *If B is an R-submodule of $A \subseteq M$, then $[A, B]$ is an R-submodule of A.*

PROOF. By definition, $[A, B]$ is an R-subgroup of M. Since B is an R-submodule of A, $r(a+b)-ra \in B$ so $[a, b, r] \in B \subset A$ and $[A, B]$ is an R-subgroup of A. Finally

$$\begin{aligned} r(a+[a_1,b,s])-ra &= r(a+b_1)-ra, \quad \text{where } [a_1,b,s]=b_1\in B \\ &= r(a+b_1)-ra-rb_1+rb_1 \\ &= [a,b_1,r]+r[a_1,b,s]\in[A,B] \end{aligned}$$

for all $a\in A$, $r\in R$ so the condition (SM) holds.

Define $M'_R=[M,M]_R$. By the proposition, M' is an R-submodule of M, and in fact R' is an ideal of R. Define M to be central if $M=Z_R(M)$ (iff $M'=0$).

THEOREM 2.3. (a) *M/M' is central and if A is an R-submodule of M, M/A is central iff $A\supseteq M'$.*

(b) *If N is an R-subgroup of M and $N\supseteq M'$, then N is an R-submodule.*

PROOF. (a) $A\supseteq M'$ iff $[x,y,r]\in A$ for all $r\in R$, $x,y\in M$ iff $[x+A,y+A,r]=0$ in M/A for all $r\in R$, $x,y\in M$ iff $[M/A,M/A]=0$.

(b) N/M' is an R-subgroup of M/M' by the isomorphism theorem and M/M' is central. Therefore N/M' is an R-submodule of M/M' and so N is an R-submodule of M.

REMARK 1. Since M is unital we see M is central iff it is abelian and R is distributive over M.

REMARK 2. In general $[M,M]_Z\neq[M,M]_R$. For example if G is an abelian group, G is a module over the near-ring $R=\{\text{maps } f:G\to G \mid f(0)=0\}$. Clearly $[G,G]_Z=0$, but $[G,G]_R=D(G)$ the distributor submodule which is nonzero as R is not distributive over G.

Define as usual Ann $M=\{r\in R \mid rM=0\}$. Then Ann M is an ideal in R. If Ann $M=0$, call M faithful.

PROPOSITION 2.4. *If there exists a faithful central R-module M, then R is a ring.*

PROOF. Since $(r+s)m=rm+sm=sm+rm=(s+r)m$ for all $m\in M$ and similarly $r(s+t)-(rs+rt)\in$ Ann $M=0$, therefore R is abelian and distributive and so is a ring.

Define the lower central series inductively by $Z^{(0)}=M$, $Z^{(i)}=[M,Z^{(i-1)}]_R$. By Proposition 2.2 each $Z^{(i)}$ is an R-submodule of M.

THEOREM 2.5. *M is R-nilpotent of class n iff n is the least integer such that $Z^{(n)}=0$.*

PROOF. We first prove $Z^{(i)}\subseteq Z_{n-i}$. Inductively, since $Z^{(0)}=Z_n=M$, suppose $Z^{(i-1)}\subseteq Z_{n-i+1}$. Then $Z_{n-i}\supseteq[M,Z_{n-i+1}]\supseteq[M,Z^{(i-1)}]=Z^{(i)}$. Hence

$Z^{(n)} \subseteq Z_0 = 0$. Moreover n is minimal for if $Z^{(n-1)} = 0$ then using the series $Z^{(n-1)} \subset \cdots \subset Z^{(0)}$ we can show as above that $Z^{(i)} \subseteq Z_{n-1-i}$. Therefore $M = Z^{(0)} \subseteq Z_{n-1}$ which contradicts the minimality of n in the upper central series. The converse is proved similarly.

3. **Solvability.** Define the derived series for M inductively by $M^{(1)} = [M, M]_R$, $M^{(i)} = [M^{(i-1)}, M^{(i-1)}]_R$, and define M to be R-solvable if $M^{(n)} = 0$ for some n. Since inductively $M^{(i)} \subset Z^{(i)} \Rightarrow M^{(i+1)} = [M^{(i)}, M^{(i)}] \subseteq [M, Z^{(i)}] = Z^{(i+1)}$ it follows that if M is R-nilpotent, it is R-solvable.

It is clear from the definitions that if M is R-nilpotent (R-solvable) it is nilpotent (solvable) as a group. In fact if $f: S \to R$ is a near-ring homomorphism and ${}_R M$ is canonically an S-module then R-nilpotence (R-solvability) implies S-nilpotence (S-solvability).

THEOREM 3.1. *M is R-solvable iff M has a normal series whose factors are all central.*

PROOF. If M is R-solvable then $M^{(n)} = 0$, so the series $\{M^{(i)}\}$ is a normal series in view of Proposition 2.2. Moreover each factor is central by Theorem 2.3. Conversely, suppose $M \supset M_1 \supset \cdots \supset M_n = 0$ is a normal series for which M_i/M_{i+1} is central for all i. By induction if $M^{(i)} \subset M_i$, then $M^{(i+1)} = [M^{(i)}, M^{(i)}] \subset [M_i, M_i] \subset M_{i+1}$ by Theorem 2.3. Therefore $M_n = 0 \Rightarrow M^{(n)} = 0$.

PROPOSITION 3.2. (a) *Every R-subgroup A and every factor module M/B of an R-solvable module M is R-solvable.*

(b) *If B is an R-solvable R-submodule of M and M/B is R-solvable then M is R-solvable.*

PROOF. (a) The canonical monomorphism $\alpha: A \to M$ and epimorphism $\beta: M \to M/B$ induce respectively monomorphisms $\alpha^{(k)}: A^{(k)} \to M^{(k)}$ and epimorphisms $\beta^{(k)}: M^{(k)} \to (M/B)^{(k)}$.

(b) Given $0 \to B \to^{\alpha} M \to^{\pi} M/B \to 0$, $(M/B)^{(k)} = 0$ implies π restricts to $\pi^{(k)}: M^{(k)} \to 0$ so $M^{(k)} \subset \ker \pi = \operatorname{Im} \alpha$. Since $B^{(m)} = 0$, $(M^{(k)})^{(m)} = M^{(k+m)} = 0$.

4. **The distributively generated case.** In this section we shall assume that R is distributively generated (d.g.) over M, by which we mean that there exists a multiplicative semigroup $S \subseteq R$ such that $s(m+n) = sm + sn$ for all $s \in S$, $m, n \in M$ and such that S additively generates R. In this case we note that $s(-m) = -sm$ for all $s \in S$, $m \in M$ and also a normal R-subgroup of M is an R-submodule. Clearly $Z_R(M) \subseteq \{a \mid b + sa = sa + b$ for all $s \in R$, $b \in M\}$ and, when R is d.g. over M, equality holds. For suppose $r \in R$ and $r = t_1 + t_2$ where the t_i are distributive over M (the proof goes inductively for $r = \sum_{i=1}^{n} t_i$). Then if $b + sa = sa + b$ for all $b \in M$,

$s \in R$ we have

$$r(b+sa) = t_1(b+sa)+t_2(b+sa) = t_1b+t_1sa+t_2b+t_2sa$$
$$= t_1b+t_2b+t_1sa+t_2sa = rb+rsa = rsa+rb,$$

so $a \in Z(M)$.

Thus $Z_R(M)=M$ iff $Z_Z(M)=M$ so that M is R-nilpotent of class 1 iff M is nilpotent of class 1. In fact we shall show that R-nilpotence (R-solvability) is equivalent to Z-nilpotence (Z-solvability). The main results depend on the following group-theoretic lemma.

LEMMA 4.1. *Let G be a group (written multiplicatively) and let A, B be normal subgroups. Then for every integer n, for all $a_i \in A$, $b_i \in B$*

$$\left(\prod_1^n b_ia_i\right)\left(\prod_1^n b_i\right)^{-1}\left(\prod_1^n a_i\right)^{-1} \in [A, B].$$

PROOF. First note that for all $g \in G$, $a \in A$, $b \in B$ $g[a,b]g^{-1} = [gag^{-1}, gbg^{-1}] \in [A, B]$. Then for all $a_1 \in A$, $b_1 \in B$

$$(*) \qquad a_1b_1[a,b]a_1^{-1}b_1^{-1} = [a_1, b_1]b_1a_1[a,b](b_1a_1)^{-1} \in [A, B].$$

For $n=2$,

$$b_1a_1b_2a_2b_2^{-1}b_1^{-1}a_2^{-1}a_1^{-1} = b_1a_1[b_2, a_2][a_2, b_1^{-1}]b_1^{-1}a_1^{-1} \in [A, B].$$

For $n \geqq 3$ the result comes from repeated application of (*) and the identity

$$\left(\prod_1^n b_i\right)^{-1}\left(\prod_1^n a_i\right)^{-1} = \left(\prod_{i=n}^3 b_i^{-1}a_i^{-1}\left[a_i, \prod_{i-1}^2 b_j^{-1}\right]\right)b_2^{-1}a_2^{-1}\left[\prod_2^n a_i, b_1^{-1}\right]b_1^{-1}a_1^{-1}.$$

THEOREM 4.2. $[A, B]_R = [A, B]_Z$ *if A and B are R-submodules of M.*

PROOF. Since $1 \in R$, every generator of $[A, B]_Z$ is in $[A, B]_R$. Conversely if $y = r(a+b)-ra-rb$ is a generator of $[A, B]_R$ and $r = \sum_1^n s_i$, $s_i \in S$ then

$$y = \sum_{i=1}^n (s_ia+s_ib) - \left(\sum_1^n s_ia\right) - \left(\sum_1^n s_ib\right) \in [A, B]_Z$$

by the (additive form of the) lemma. Since $[A, B]_Z$ is an R-subgroup in the d.g. case, the result follows.

COROLLARY 1. *If A and B are R-submodules of M, $[A, B]_R = [B, A]_R$ and this is an R-submodule of M.*

COROLLARY 2. *M is R-nilpotent (R-solvable) iff M is Z-nilpotent (Z-solvable).*

COROLLARY 3 [3, Theorem 4.4.3]. *R abelian and d.g.$\Rightarrow R$ is a ring.*

R-solvability can be expressed in terms of an R-composition series as follows. Following [**4**] we define an ideal P of R to be prime if whenever A, B are ideals such that $AB \subseteq P$ then $A \subseteq P$ or $B \subseteq P$ (here AB refers to the additive group generated by all ab).

PROPOSITION 4.3 [**4**]. *If M is a cyclic simple module*, Ann M *is a prime ideal.*

PROPOSITION 4.4. *If M is R-solvable and has a composition series, the series has cyclic factors A_i where* Ann A_i *is a prime ideal.*

PROOF. M has a normal series with central factors. By Theorem 2.3 this can be refined to a composition series with central simple factors i.e. central irreducible factors which are therefore cyclic and so have prime annihilators by 4.3.

We shall now investigate some consequences of R-nilpotence.

Let A be an R-subgroup of M. Define the R-normalizer of A to be $N_R(A)=N(A)=\{x \in M \mid rx+a-rx \in A$ for all $a \in A$, $r \in R\}$.

PROPOSITION 4.5. (a) *$N(A)$ is an R-subgroup of M.* (b) *$N(A)$ is the largest R-subgroup of M in which A is an R-submodule.*

PROOF. (a) Consider, for $x, y \in N(A)$, $z=r(x-y)+a-r(x-y)$ where $r=\sum_1^n s_i$, $s_i \in S$. By induction on n,

$$\begin{aligned} z &= s_1x - s_1y + \left(\sum_2^n s_i\right)(x-y) + a - \left(\sum_2^n s_i\right)(x-y) + s_1y - s_1x \\ &= s_1x - s_1y + a' + s_1y - s_1x, \qquad a' \in A, \end{aligned}$$

so $z \in A$. Clearly $x \in N(A) \Rightarrow tx \in N(A)$ for all $t \in R$.

(b) A is clearly a normal subgroup of $N(A)$ so it is an R-submodule since M is d.g. If A is an R-submodule of $B \subset M$ then, for all $b \in B$, $r \in R$, $rb \in B$; so by the normality of A in B, $rb+a-rb \in B$ for all $a \in A$ and hence $b \in N(A)$, i.e. $B \subseteq N(A)$.

PROPOSITION 4.6. *If M is nilpotent and A is an R-subgroup of M then $A \subsetneqq N(A)$.*

PROOF. If k is the largest integer such that $Z_k \subset A$, choose $x \in Z_{k+1}$, $x \notin A$. Then for all $a \in A$, $r \in R$, $a+[-a, x, 1, r]=rx+a-rx \in A+Z_k \subset A$ so $x \in N(A)$.

COROLLARY. *If M is nilpotent, every maximal R-subgroup of M is an R-submodule.*

LEMMA 4.7. *A cyclic R-module Rm is central iff $R' \subset$* Ann m.

PROOF. Rm is central iff $x+sy=sy+x$ for all $x, y \in Rm$, $s \in R$ iff $(r+st)m=(st+r)m$ for all $r, s, t \in R$ iff $r+st-(st+r) \in \operatorname{Ann} m$ iff $[R, R] \subset \operatorname{Ann} m$.

Define the Frattini subgroup of M to be

$$\begin{aligned} F(M) = F_R(M) &= \bigcap \{\text{maximal proper } R\text{-subgroups of } M\} \quad \text{if any exist} \\ &= M \quad \text{otherwise.} \end{aligned}$$

Thus by universal algebra, $F(M)$ is the set of nongenerators of M.

PROPOSITION 4.8. *If $F(M) \supset M'$, every maximal R-subgroup of M is an R-submodule and when $R' \subset \operatorname{Ann} M$ the converse is true.*

PROOF. $F(M) \supset M'$ implies $A \supset M'$ for every maximal R-subgroup A. By Theorem 2.3, A is an R-submodule. Conversely if every maximal R-subgroup A is an R-submodule then M/A is an irreducible R-module so is cyclic. Writing $M/A=Ra$, $R' \subset \operatorname{Ann} M \subset \operatorname{Ann} (M/A) \subset \operatorname{Ann} a$ implies by the lemma that M/A is central. Thus $A \supset M'$ and as this is true for all maximal R-subgroups A, $F(M) \supset M'$ as required.

THEOREM 4.9. *If M is nilpotent and $R' \subset \operatorname{Ann} M$ then $M' \subset F(M)$.*

PROOF. If M is nilpotent, every maximal R-subgroup is an R-submodule by the corollary to 4.6 so $M' \subset F(M)$ by Proposition 4.8.

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