PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, October 1973

CLASSIFICATION OF BOUNDED SOLUTIONS OF A LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATION[1]

THOMAS G. HALLAM

ABSTRACT. An elementary criterion, depending only upon the initial data of a solution, is formulated to determine the boundedness of solutions of a nonhomogeneous linear system of ordinary differential equations. The associated homogeneous linear differential equation is required to be either conditionally stable or conditionally asymptotically stable.

1. **Introduction.** Substantive information for linear perturbation problems includes a knowledge of the behavior of the solutions of the nonhomogeneous linear differential equation

$$x' = A(t)x + f(t) \tag{1}$$

whenever the associated homogeneous system

$$y' = A(t)y \tag{2}$$

possesses a prescribed property. In this note, we assume that the homogeneous equation (2) possesses a certain conditional stability property; then, a classification of the bounded solutions of (1) is obtained. The boundedness criterion is phrased solely in terms of the initial value of the solution.

2. **Hypotheses for (1) and (2).** In equations (1) and (2), x, y are elements in an n-dimensional vector space X; $A(t)$ is a continuous $n\times n$ matrix defined on $R=(-\infty, \infty)$; and, $f\in C[R, X]$. Let $Y(t)$ denote the fundamental matrix of (1) that satisfies the condition $Y(0)=I_n$, I_n is the $n\times n$ identity matrix. The symbol $|\cdot|$ denotes a norm on X as well as a corresponding (consistent) matrix norm.

Received by the editors October 18, 1972 and, in revised form, January 15, 1973.
AMS (MOS) subject classifications (1970). Primary 34A30, 34C10, 34D20.
Key words and phrases. Conditional stability, conditional asymptotic stability, variation of parameters, bounded solutions.
[1] This research was supported in part by the National Science Foundation under Grant GP-11543.

3. **The conditional asymptotic stability case.** A fundamental requirement for (2) in this section is

(H.1) Let there exist supplementary projections P_0, P_{-1}, P_1, P_∞ and constants K, q with $K>0$ and $1\leqq q<\infty$ such that

$$\left[\int_{-\infty}^{t}|Y(t)P_{-1}Y^{-1}(s)|^q\,ds\right]^{1/q}+\left|\int_0^t|Y(t)P_0Y^{-1}(s)|^q\,ds\right|^{1/q}$$
$$+\left[\int_t^\infty|Y(t)P_1Y^{-1}(s)|^q\,ds\right]^{1/q}\leqq K,\qquad t\in R;$$
$$\left[\int_t^\infty|Y(t)P_\infty Y^{-1}(s)|^q\,ds\right]^{1/q}\leqq K,\qquad t\in R_+\equiv[0,\infty);$$
$$\left[\int_{-\infty}^t|Y(t)P_\infty Y^{-1}(s)|^q\,ds\right]^{1/q}\leqq K,\qquad t\in R_-\equiv(-\infty,0].$$

A complementary requirement on (1) is

(H.2) The nonhomogeneous term f is in $L^p(R)$ where $1/p+1/q=1$. The norm of $f\in L^p(R)$ is denoted by $|f|_p$.

THEOREM 1. *Let conditions* (H.1) *and* (H.2) *be satisfied for equations* (1) *and* (2). *Then, the following conclusions hold*:

(i) *A solution $x=x(t)$ of* (1) *is bounded on R_+ if and only if*

$$\begin{aligned}P_1x(0)&=-\int_0^\infty P_1Y^{-1}(s)f(s)\,ds;\quad \textit{and}\\ P_\infty x(0)&=-\int_0^\infty P_\infty Y^{-1}(s)f(s)\,ds.\end{aligned}\tag{3}$$

(ii) *A solution $x=x(t)$ of* (2) *is bounded on R_- if and only if*

$$\begin{aligned}P_{-1}x(0)&=\int_{-\infty}^0 P_{-1}Y^{-1}(s)f(s)\,ds;\quad \textit{and}\\ P_\infty x(0)&=\int_{-\infty}^0 P_\infty Y^{-1}(s)f(s)\,ds.\end{aligned}\tag{4}$$

(iii) *A solution $x=x(t)$ of* (2) *is bounded on R if and only if both* (3) *and* (4) *hold.*

PROOF. The hypothesis (H.1) used in conjunction with the Lemmas of [**4**] (also see R. Conti [**1**] and W. A. Coppel [**2**, pp. 68, 74]) imply

$$\lim_{|t|\to\infty}|Y(t)P_0|=0;\tag{5}$$

$$\lim_{t\to\infty}|Y(t)P_{-1}|=0\quad\text{and}\tag{6}$$
$$\limsup_{t\to-\infty}|Y(t)P_{-1}\xi|=\infty\quad\text{provided } P_{-1}\xi\neq0;$$

$$\lim_{t\to-\infty} |Y(t)P|_1 = 0 \quad \text{and} \tag{7}$$

$$\limsup_{t\to\infty} |Y(t)P_1\xi| = \infty \quad \text{provided } P_1\xi \neq 0;$$

$$\limsup_{|t|\to\infty} |Y(t)P_\infty\xi| = \infty \quad \text{provided } P_\infty\xi \neq 0. \tag{8}$$

It follows from (5), (6), and (7) that there is a constant $M>0$ such that

$$|Y(t)P_{-1}| + |Y(t)P_0| \leqq M, \qquad t \in R_+; \quad \text{and}$$

$$|Y(t)P_1| + |Y(t)P_0| \leqq M, \qquad t \in R_-.$$

Proceeding formally, we use the variation of parameters formula to obtain for a solution x of (1) valid on R_+,

$$\begin{aligned} x(t) &= Y(t)x(0) + \int_0^t Y(t)Y^{-1}(s)f(s)\,ds \\ &= Y(t)[P_0 + P_{-1} + P_1 + P_\infty]x(0) + \int_{-\infty}^t Y(t)P_{-1}Y^{-1}(s)f(s)\,ds \\ &\quad + \int_0^t Y(t)P_0Y^{-1}(s)f(s)\,ds - \int_t^\infty Y(t)[P_1 + P_\infty]Y^{-1}(s)f(s)\,ds \\ &\quad + \int_0^\infty Y(t)[P_1 + P_\infty]Y^{-1}(s)f(s)\,ds \\ &\quad - \int_{-\infty}^0 Y(t)P_{-1}Y^{-1}(s)f(s)\,ds, \qquad t \in R_+. \end{aligned} \tag{9}$$

For $t \in R_-$, we have

$$\begin{aligned} x(t) &= Y(t)[P_0 + P_{-1} + P_1 + P_\infty]x(0) \\ &\quad + \int_{-\infty}^t Y(t)[P_{-1} + P_\infty]Y^{-1}(s)f(s)\,ds \\ &\quad + \int_0^t Y(t)P_0Y^{-1}(s)f(s)\,ds - \int_t^\infty Y(t)P_1Y^{-1}(s)f(s)\,ds \\ &\quad - \int_{-\infty}^0 Y(t)[P_{-1} + P_\infty]Y^{-1}(s)f(s)\,ds + \int_0^\infty Y(t)P_1Y^{-1}(s)f(s)\,ds. \end{aligned} \tag{10}$$

It is a direct consequence of Hölder's inequality, hypotheses (H.1) and (H.2) that each of the above integrals, with the exception of the last ones, in (9) and (10) is well defined. To verify that the last integrals are also well defined, we evaluate an equality in (H.1) at $t=0$ to obtain

$$\left[\int_{-\infty}^0 |P_{-1}Y^{-1}(s)|^q\,ds\right]^{1/q} + \left[\int_0^\infty |P_1Y^{-1}(s)|^q\,ds\right]^{1/q} \leqq K. \tag{11}$$

From (11) and (H.2), we have

$$\begin{aligned}(12)\quad &\left|\int_{-\infty}^{0} Y(t)P_{-1}Y^{-1}(s)f(s)\,ds\right| \\ &\qquad \leqq |Y(t)P_{-1}|\left[\int_{-\infty}^{0}|P_{-1}Y^{-1}(s)|^{q}\,ds\right]^{1/q}|f|_{p} \leqq MK\,|f|_{p}.\end{aligned}$$

The same bound also holds for the last integral in (10); hence, $x=x(t)$ is well defined. A direct calculation shows that the function $x=x(t)$ with domain R defined by (9) for $t\in R_{+}$ and by (10) for $t\in R_{-}$ is the solution of (1) that passes through the initial position $x(0)$.

We first verify that assertion (i) holds. Since x is given by (9) for $t\in R_{+}$, an application of Hölder's inequality, conditions (H.1), (H.2) and inequality (12) leads to

$$\begin{aligned}(13)\quad &\left|x(t)-Y(t)[P_{1}+P_{\infty}]\left[x(0)+\int_{0}^{\infty}Y^{-1}(s)f(s)\,ds\right]\right| \\ &\qquad \leqq M\,|[P_{-1}+P_{0}]x(0)|+(2+M)K\,|f|_{p}, \qquad t\in R_{+}.\end{aligned}$$

From (13), we note that the statement—$x(t)$ is bounded on R_{+}—is equivalent to the statement—

$$(14)\qquad Y(t)[P_{1}+P_{\infty}]\left[x(0)+\int_{0}^{\infty}Y^{-1}(s)f(s)\,ds\right]$$

is bounded on R_{+}. Equations (7) and (8) show that the expression in (14) is bounded on R_{+} if and only if (3) holds; this proves (i).

Conclusion (ii) is verified in an analogous fashion using equation (10). The analogue of inequality (13) is

$$\begin{aligned}&\left|x(t)-Y(t)[P_{-1}+P_{\infty}]\left[x(0)-\int_{-\infty}^{0}Y^{-1}(s)f(s)\,ds\right]\right| \\ &\qquad \leqq M\,|[P_{1}+P_{0}]x(0)|+(2+M)K\,|f|_{p}, \qquad t\in R_{-}.\end{aligned}$$

This inequality coupled with equations (6) and (8) leads to (ii).

The assertion (iii) is an immediate consequence of (i) and (ii).

REMARK 1. As a corollary to (iii), we find that a necessary condition for (2) to possess a solution that is bounded on R is that the forcing function f satisfy the equation

$$(15)\qquad \int_{-\infty}^{\infty}P_{\infty}Y^{-1}(s)f(s)\,ds=0.$$

If it is known that (2) has a bounded solution for every $f\in L^{p}$ (see [**1**] for this type of result on R_{+}) then, from (15), we have $P_{\infty}\equiv 0$. This extends a recent result of D. L. Lovelady [**6**] who considered the case $p=\infty$.

REMARK 2. If the conditional asymptotic stability hypothesis (H.1) is replaced by the corresponding conditional stability analogue then it is no longer necessary that an L^∞-decomposition of the solution space is effected. However, Theorem 1 still holds if it is assumed that (H.1) and (H.2) are replaced by the corresponding requirements in L^∞ and L^1 respectively and, in addition, that

$$\limsup_{t\to-\infty} |Y(t)P_{-1}\xi| = \infty \quad \text{provided } P_{-1}\xi \neq 0,$$

$$\limsup_{t\to\infty} |Y(t)P_{1}\xi| = \infty \quad \text{provided } P_{1}\xi \neq 0, \quad \text{and}$$

$$\limsup_{|t|\to\infty} |Y(t)P_{\infty}\xi| = \infty \quad \text{provided } P_{\infty}\xi \neq 0.$$

With these assumptions, the proof parallels that of the case $1 \leq q < \infty$.

4. **Concluding remarks.** The projection P_∞ has not been frequently utilized in connection with perturbation problems. Most research efforts in this direction have been devoted to finding bounded solutions of differential equations; hence, P_∞ has been purposefully neglected.

In [5], a discussion of the unbounded solutions of a functional perturbation problem is presented under conditions where P_∞ need not be zero. A different integral equation representation and hypothesis on P_∞ is used there.

Information about the behavior of the solutions of (2) is necessary before the above theorems can be applied. J. Macki and J. Muldowney [7] discuss the bounded and unbounded solutions of (2). Other information about the bounded solutions of (2) can be found in [3].

We conclude this note with a simple but illustrative example. Let the vector y be given by $y = \mathrm{col}(y_1, y_2, y_3, y_4)$ and the matrix A be given by $A(t) = \mathrm{diag}(-2t, -1, 1, 2t)$. The fundamental matrix Y of this system (2) with $Y(0) = I_4$ is $Y(t) = \mathrm{diag}(e^{-t^2}, e^{-t}, e^t, e^{t^2})$. The projections P_i of condition (H.1) are taken to be

$$P_{-1} = \mathrm{diag}(0, 1, 0, 0); \qquad P_0 = \mathrm{diag}(1, 0, 0, 0);$$

$$P_{1} = \mathrm{diag}(0, 0, 1, 0); \qquad P_\infty = \mathrm{diag}(0, 0, 0, 1).$$

The matrices $Y(t)P_iY^{-1}(s)$ are given by

$$Y(t)P_{-1}Y^{-1}(s) = \mathrm{diag}(0, \exp[s - t], 0, 0);$$

$$Y(t)P_{0}Y^{-1}(s) = \mathrm{diag}(\exp[s^2 - t^2], 0, 0, 0);$$

$$Y(t)P_{1}Y^{-1}(s) = \mathrm{diag}(0, 0, \exp[t - s], 0);$$

$$Y(t)P_{\infty}Y^{-1}(s) = \mathrm{diag}(0, 0, 0, \exp[t^2 - s^2]).$$

The hypothesis (H.1) is satisfied for $q=1$. Thus, if $f=\operatorname{col}(f_1, f_2, f_3, f_4)$ is bounded and continuous on R, then Theorem 1 leads to the following criteria for the determination of the boundedness of solutions of (1).

A solution $x(t)=\operatorname{col}(x_1(t), x_2(t), x_3(t), x_4(t))$ of (1) is bounded on R_+ if and only if $x_3(0)=-\int_0^\infty e^{-s} f_3(s)\,ds$ and $x_4(0)=-\int_0^\infty e^{-s^2} f_4(s)\,ds$. A solution x is bounded on R_- if and only if $x_2(0)=\int_{-\infty}^0 e^{s} f_2(s)\,ds$ and $x_4(0)=\int_{-\infty}^0 e^{-s^2} f_4(s)\,ds$.

References

1. R. Conti, *On the boundedness of solutions of ordinary differential equations*, Funkcial. Ekvac. **9** (1966), 23–26. MR **37** #3102.

2. W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, Boston, Mass., 1965. MR **32** #7875.

3. T. G. Hallam, *An asymptotic solution of a nonhomogeneous linear system of differential equations*, Proc. Amer. Math. Soc. **29** (1971), 529–534. MR **43** #3561.

4. ———, *On nonlinear functional perturbation problems for ordinary differential equations*, J. Differential Equations **12** (1972), 63–80.

5. ———, *On representation and boundedness of solutions of a functional differential equation*, Math. Systems Theory (to appear).

6. D. L. Lovelady, *Boundedness and ordinary differential equations on the whole line* (to appear).

7. J. W. Macki and J. S. Muldowney, *The asymptotic behavior of solutions to linear systems of ordinary differential equations*, Pacific J. Math. **33** (1960), 693–706. MR **42** #3360.

Department of Mathematics, Florida State University, Tallahassee, Florida 32306