ISOMORPHISM THEOREMS FOR INFINITE ORDER DIFFERENTIAL OPERATORS

J. D. BUCKHOLTZ

ABSTRACT. The operators studied are infinite order linear differential operators with constant coefficients. We require that the characteristic function of the operator be analytic in a neighborhood of 0 and have a radius of convergence which is less than that of its reciprocal. It is shown that such operators may be regarded as isomorphisms between Banach spaces of entire functions. These Banach spaces, in every case, are isomorphic to the sequence space c_0 . Further, translation in the domain plane defines an automorphism on both the domain and range space of the operator.

Let $\varphi(z) = \sum_{j=0}^{\infty} b_j z^j$ have radius of convergence r ($0 < r < \infty$) and suppose that $\theta(z) = 1/\varphi(z) = \sum_{j=0}^{\infty} a_j z^j$ has radius of convergence greater than r . Let D denote the derivative operator and denote by $\varphi(D)$ the operator which takes an analytic function u into

$$(1) \quad \varphi(D)u = \sum_{j=0}^{\infty} b_j u^{(j)}.$$

It is natural to take for the domain of $\varphi(D)$ the family of entire functions u such that (1) is uniformly convergent on every compact set [3, Lemma 1.1]. Subject to this requirement on its domain, the behavior of the operator $\varphi(D)$ can be completely determined. Under suitable norms, the domain and range of $\varphi(D)$ are Banach spaces of entire functions, and $\varphi(D)$ is an isomorphism (i.e., a linear homeomorphism) between these spaces. An explicit determination of these spaces was obtained in [1], where it was shown that the spaces are completely determined by the singularities of φ on the circle $|z|=r$. The basic structure of the spaces is, however, independent of the function φ . In every case, they are isomorphic to c_0 , the space of complex sequences with limit 0, normed with the supremum norm. The proof given in [1] for this result relied heavily on infinite matrices and to some extent on the Appell polynomials generated by φ . The purpose of the present paper is to give a somewhat altered proof which avoids these difficulties.

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Since $\theta=1/\varphi$ is analytic in the closed disk $|z|\leq r$, it follows that φ is zero-free in the closed disk $|z|\leq r$, and that its singularities on $|z|=r$ are poles. Let α_q^{-1} , $1\leq q\leq\lambda$, denote the (distinct) poles of φ on $|z|=r$, and let $m(q)$ denote the order of the pole α_q^{-1} . Set $m=\max m(q)$, $1\leq q\leq\lambda$. It was shown in [3] that the domain of $\varphi(D)$ is the space $\mathcal{U}$ of entire functions u such that each of the series

$$\sum_{k=0}^{\infty} \binom{k+m(q)-1}{m(q)-1} \alpha_q^k u^{(k)}(0), \quad 1\leq q\leq\lambda,$$

converges. For $u\in\mathcal{U}$ we set

$$\|u\|_0 = \max_{1\leq q\leq\lambda} \sup_{0\leq n<\infty} \left| \sum_{k=n}^{\infty} \binom{k+m(q)-1}{m(q)-1} \alpha_q^k u^{(k)}(0) \right|.$$

Let $Q(z)=\prod_{q=1}^{\lambda} (1-\alpha_q z)^{\min\{m(q), m-1\}}$ and let $\mathcal{F}$ denote the space of entire functions f such that

$$\lim_{n\rightarrow\infty} r^{-n} (D^n f)(0) = \lim_{n\rightarrow\infty} n^{m-1} r^{-n} (D^n Q(D)f)(0) = 0.$$

For $f\in\mathcal{F}$, set

$$\|f\|_1 = \sup_{0\leq n<\infty} \left(\frac{1}{2} \right) \left\{ r^{-n} |(D^n f)(0)| + \binom{n+m-1}{m-1} r^{-n} |(D^n Q(D)f)(0)| \right\}.$$

It was shown in [3] that $\mathcal{F}$ is the range of $\varphi(D)$. The following result is an immediate consequence of Theorem 1.5 of [3] and the Mapping Theorem of [1].

THEOREM 1. *The operator $\varphi(D)$ has domain $\mathcal{U}$ and range $\mathcal{F}$. Further, $\varphi(D)$ is 1-1, bounded, and possesses a bounded inverse. Its inverse is the restriction to $\mathcal{F}$ of the operator $\theta(D)$.*

The operator $\varphi(D)$ defines an isomorphism between $\mathcal{U}$ and $\mathcal{F}$, and this is the principal tool needed to show that $\mathcal{F}$ is isomorphic to c_0 . Let E_r denote the space of all entire f such that $f^{(n)}(0)=o(r^n)$, $n\rightarrow\infty$, and, for $f\in E_r$, set

$$\|f\|_2 = \sup_{0\leq n<\infty} |r^{-n} f^{(n)}(0)|.$$

The identification of f with the sequence $\{r^{-n} f^{(n)}(0)\}_0^{\infty}$ defines a linear isometry between E_r and c_0 ; we shall use differential operators to construct an isomorphic map of $\mathcal{F}$ onto E_r .

Apart from Theorem 1, the construction depends on three observations about the spaces $\mathcal{F}$ and $\mathcal{U}$. First, note that if $m=1$ (φ has only simple poles on $|z|=r$), then $Q(z)\equiv 1$ and $\mathcal{F}$ is the space E_r . The second observation is slightly deeper and crucial to the argument: the polynomial Q ,

and therefore the space $\mathcal{F}$, can not distinguish between a pole of φ of order m and a pole of order $m-1$. In particular, if we let

$$v_1(z) = \theta(z) \prod_{m(q)=m-1} (1 - \alpha_q z), \quad \mu_1(z) = \frac{1}{v_1(z)},$$

and let $\mathcal{F}(\mu_1)$ denote the space obtained from $\mathcal{F}$ by replacing φ by μ_1 , then $\mathcal{F}(\mu_1) = \mathcal{F}$. Since we shall have to consider several spaces, we shall sometimes write $\mathcal{F}(\varphi)$ and $\mathcal{U}(\varphi)$ for $\mathcal{F}$ and $\mathcal{U}$ and use $\mathcal{F}(\mu)$ and $\mathcal{U}(\mu)$ to denote the spaces obtained by replacing φ by a function μ . The norm on $\mathcal{U}(\mu)$ which corresponds to $\|\cdot\|_0$ will be denoted by $N_0(\cdot; \mu)$, and the norm on $\mathcal{F}(\mu)$ which corresponds to $\|\cdot\|_1$ will be denoted by $N_1(\cdot; \mu)$. In addition to μ_1 and v_1 , we shall also need the functions

$$v_2(z) = \prod_{m(q) \geq m-1} (1 - a_q z), \quad \mu_2(z) = \frac{1}{v_2(z)}, \quad \text{and} \quad \mu_3(z) = \{\mu_2(z)\}^m.$$

Our third observation concerns the space $\mathcal{U}$. It is possible to replace the norm $\|\cdot\|_0$ by an equivalent norm $\|\cdot\|_3$ which depends only on the poles of φ of orders m and $m-1$. The convergence of

$$(2) \quad \sum_{k=0}^{\infty} \binom{k + m(q) - 1}{m(q) - 1} \alpha_q^k u^{(k)}(0)$$

for values of q for which $m(q)=m$ obviously implies the convergence of (2) for values of q such that $m(q) \leq m-2$. Therefore $\mathcal{U}$ is the space of all entire u such that (2) converges for all q such that $m(q) \geq m-1$. It is not hard to show that the norm

$$\|u\|_3 = \max_{m(q) \geq m-1} \sup_{0 \leq n < \infty} \left| \sum_{k=n}^{\infty} \binom{k + m(q) - 1}{m(q) - 1} \alpha_q^k u^{(k)}(0) \right|$$

is equivalent to the norm $\|\cdot\|_0$ (a proof of this is contained in [1, Lemma 4.1]). In the space $\mathcal{U}(\mu)$ we shall write $N_3(\cdot; \mu)$ for the norm corresponding to $\|\cdot\|_3$.

Let I denote the identity operator and let Γ_{m-1} denote the operator which transforms an entire function f into

$$(\Gamma_{m-1}f)(z) = \sum_{k=0}^{\infty} \binom{k + m - 1}{m - 1} f^{(k)}(0) \frac{z^k}{k!}.$$

From the factorization

$$\Gamma_{m-1} = \frac{1}{(m-1)!} \prod_{t=1}^{m-1} (zD + tI),$$

one sees that Γ_{m-1} is a differential operator of order $m-1$ with polynomial coefficients.

We are now in a position to establish an isomorphism between $\mathcal{F}$ and E_r . We have seen that $\mathcal{F} = \mathcal{F}(\varphi)$ is the same space as $\mathcal{F}(\mu_1)$. Theorem 1 applied to the operator $\mu_1(D)$ establishes an isomorphism between $\mathcal{F}(\mu_1)$ and $\mathcal{U}(\mu_1)$; specifically, $\nu_1(D)$ is an isomorphic mapping of $\mathcal{F}$ onto $\mathcal{U}(\mu_1)$.

We now show that the identity transformation defines an isomorphism between $\mathcal{U}(\mu_1)$ and $\mathcal{U}(\mu_3)$. Note that μ_1 has no poles of order $m-1$ on $|z|=r$ and the poles of μ_1 of order m on $|z|=r$ are the poles of φ of orders m and $m-1$ on $|z|=r$. To write down the $N_0(\ ; \mu_1)$ norm of a member of $\mathcal{U}(\mu_1)$ is fairly complicated unless one introduces new names for the poles of μ_1 . There is, however, no difficulty in writing down the equivalent norm $N_3(\ ; \mu_1)$. We have, for $u \in \mathcal{U}(\mu_1)$,

$$N_3(u; \mu_1) = \max_{m(q) \geq m-1} \sup_{0 \leq n < \infty} \left| \sum_{k=n}^{\infty} \binom{k+m-1}{m-1} \alpha_q^k u^{(k)}(0) \right|.$$

Note that $\mathcal{U}(\mu_1)$ and $\mathcal{U}(\mu_3)$ contain the same functions and that $N_3(u; \mu_1) = N_0(u; \mu_3)$; this justifies our assertion about the identity map.

Let V denote the image of $\mathcal{U}(\mu_3)$ under the operator Γ_{m-1} , and for functions $v = \Gamma_{m-1}u$ in V , set

$$\begin{aligned} \|v\|_4 &= \|\Gamma_{m-1}u\|_4 = N_0(u; \mu_3) \\ &= \max_{m(q) \geq m-1} \sup_{0 \leq n < \infty} \left| \sum_{k=n}^{\infty} \binom{k+m-1}{m-1} \alpha_q^k u^{(k)}(0) \right|. \end{aligned}$$

This choice of norm makes Γ_{m-1} a linear isometry and extends our isomorphism from $\mathcal{F}$ to V . For $v = \Gamma_{m-1}u$ we have $v^{(k)}(0) = \binom{k+m-1}{m-1} u^{(k)}(0)$, $k=0, 1, 2, \dots$. Therefore

$$\|v\|_4 = \max_{m(q) \geq m-1} \sup_{0 \leq n < \infty} \left| \sum_{k=n}^{\infty} \alpha_q^k v^{(k)}(0) \right|.$$

Note also that $v \in V$ if and only if $\sum_{k=0}^{\infty} \alpha_q^k v^{(k)}(0)$ converges for all q such that $m(q) \geq m-1$. One sees therefore that V normed with $\|\cdot\|_4$ is the same space as $\mathcal{U}(\mu_2)$ with norm $N_0(\ ; \mu_2)$.

Theorem 1 applied to $\mu_2(D)$ asserts that $\mu_2(D)$ is an isomorphic map of $V = \mathcal{U}(\mu_2)$ onto $\mathcal{F}(\mu_2)$. But the poles of μ_2 on $|z|=r$ are all simple, so that from our first observation we obtain $\mathcal{F}(\mu_2) = E_r$. Retracing our steps, we see that the isomorphism between $\mathcal{F}$ and E_r is established by the mappings

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(\mu_1) \xrightarrow{\nu_1(D)} \mathcal{U}(\mu_1) \xrightarrow{I} \mathcal{U}(\mu_3), \\ \mathcal{U}(\mu_3) &\xrightarrow{\Gamma_{m-1}} V = \mathcal{U}(\mu_2) \xrightarrow{\mu_2(D)} \mathcal{F}(\mu_2) = E_r. \end{aligned}$$

This proves the following.

THEOREM 2. *The operator $\mu_2(D) \circ \Gamma_{m-1} \circ \nu_1(D)$ is an isomorphic map of $\mathcal{F}$ onto E_r .*

For a fixed complex number w , let T_w denote the operator which transforms an entire function f into $(T_w f)(z) = f(z+w)$. It is worth noting that the spaces $\mathcal{U}$ and $\mathcal{F}$ are translation invariant, i.e., the restriction of T_w to either space is a bijection. As one would expect, the operator T_w (restricted to either space) is bounded and has a bounded inverse.

THEOREM 3. *The operator T_w defines an automorphism on each of the spaces $\mathcal{U}$ and $\mathcal{F}$.*

PROOF. The function $\mu(z) = e^{wz}\varphi(z)$ has the same singularities as φ . Therefore $\mathcal{U}(\mu) = \mathcal{U}$, $\mathcal{F}(\mu) = \mathcal{F}$, and $\mu(D)$ is an isomorphic map from $\mathcal{U}$ to $\mathcal{F}$. Note that $T_w = \exp\{wD\}$. The proof depends on showing that $\mu(D)$ has the operator factorizations

$$(3) \quad \mu(D) = \varphi(D) \circ T_w$$

and

$$(4) \quad \mu(D) = T_w \circ \varphi(D).$$

For the time being we take these for granted. Suppose that $u \in \mathcal{U}$. From (3) we have $\mu(D)u = \varphi(D)\{T_w u\}$. Applying $\theta(D)$ to both sides of this equation, we obtain $T_w u = \theta(D)\{\mu(D)u\}$. Therefore the restriction to $\mathcal{U}$ of T_w satisfies $T_w = \theta(D) \circ \mu(D)$, and it follows from Theorem 1 that this is an automorphism.

Suppose now that $f \in \mathcal{F}$, and set $u = \theta(D)f$. From (4) and Theorem 1 we obtain $\mu(D)u = T_w\{\varphi(D)u\} = T_w f$. But $\mu(D)u = \mu(D)\{\theta(D)f\}$. Therefore $T_w f = \mu(D)\{\theta(D)f\}$, and the restriction to $\mathcal{F}$ of T_w satisfies $T_w = \mu(D) \circ \theta(D)$. As before, it follows that this restriction is an automorphism.

To establish (3) and (4), we need some information about Appell polynomials. Let $\{p_k\}_0^\infty$ and $\{\rho_k\}_0^\infty$ denote the Appell polynomial sequences generated by φ and μ respectively. It is not hard to show that $\rho_k(z) = p_k(z+w)$. We shall also need the fact [3, Theorem 2.1] that, for $u \in \mathcal{U}$, the expansions

$$\{\varphi(D)u\}(z) = \sum_{k=0}^{\infty} u^{(k)}(0)p_k(z) = \sum_{k=0}^{\infty} u^{(k)}(w)p_k(z-w)$$

and

$$\{\mu(D)u\}(z) = \sum_{k=0}^{\infty} u^{(k)}(0)\rho_k(z) = \sum_{k=0}^{\infty} u^{(k)}(w)\rho_k(z-w)$$

are valid.

For $u \in \mathcal{U}$ we have

$$\begin{aligned}\{\mu(D)u\}(z) &= \sum_{k=0}^{\infty} u^{(k)}(0)\rho_k(z) = \sum_{k=0}^{\infty} u^{(k)}(0)p_k(z+w) \\ &= \sum_{k=0}^{\infty} u^{(k)}(w)p_k(z) = \{\varphi(D)(T_w u)\}(z).\end{aligned}$$

This establishes (3). To establish (4), note that

$$\{\mu(D)u\}(z) = \sum_{k=0}^{\infty} u^{(k)}(0)\rho_k(z) = \sum_{k=0}^{\infty} u^{(k)}(0)p_k(z+w) = \{T_w\{\varphi(D)u\}\}(z).$$

Therefore $\mu(D)u = T_w\{\varphi(D)u\}$, and the proof is complete.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506