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AN INVARIANT OF CONFORMAL MAPPINGS

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ABSTRACT. A result of W. Blaschke on conformal invariants of a surface is generalized.

1. **Introduction.** A conformal mapping on euclidean m-space E^m can be decomposed into a product of similarity transformations and inversions $\{\pi_j\}$ (Haantjes [2]). Let M be a surface in E^m. If the center of inversion π_i of the conformal mapping does not lie on the surface M for all π_i, then the conformal mapping is called a conformal mapping of E^m with respect to M. A quantity on M is called a *conformal invariant* if it is invariant under conformal mappings of E^m with respect to M.

The main purpose of this note is to prove the following

THEOREM. *Let M be a surface in E^m with Gauss curvature K, mean curvature H and volume element dV. Then $(H^2-K)\,dV$ is a conformal invariant.*

If the codimension is one, this theorem was given by Blaschke [1].

2. **Proof of the Theorem.** It is obvious that the quantity $(H^2-K)\,dV$ is invariant under similarity transformations (motions and homothetics on E^m). Hence, it suffices to prove the Theorem for inversions. Let π be an inversion on E^m such that the center of π does not lie on the surface M. We choose the origin at the center of the inversion π. Let x and $\bar{x}$ be the position vectors of the origin surface M and the inverse surface $\bar{M}$ respectively, and let c be the radius of inversion π. Then we have

$$\bar{x} = (c^2/r^2)x, \qquad r^2 = x \cdot x. \tag{1}$$

From this we find that

$$d\bar{x} = (c^2/r^2)\,dx - (2c^2/r^3)(dr)x, \tag{2}$$

$$d\bar{x} \cdot d\bar{x} = (c^4/r^4)\,dx \cdot dx. \tag{3}$$

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Hence the volume element $d\bar{V}$ of $\bar{M}$ is given by

$$d\bar{V} = (c^4/r^4)\, dV. \tag{4}$$

Let $e_3, \cdots, e_{m-2}$ be any $m-2$ mutually orthogonal unit normal local vector fields on M. Then

$$\bar{e}_\alpha = (2(x \cdot e_\alpha)/r^2)x - e_\alpha, \qquad \alpha = 3, \cdots, m-2, \tag{5}$$

are $m-2$ mutually orthogonal unit normal vector fields on $\bar{M}$. From (2) and (5), we obtain

$$d\bar{x} \cdot d\bar{e}_\alpha = (2c^2(x \cdot e_\alpha)/r^4)\, dx \cdot dx - (c^2/r^2)\, dx \cdot de_\alpha. \tag{6}$$

Combining (3) and (6), we find that, for any unit vector e of M in E^m, the principal curvatures $k_i(e)$, $i=1, 2$, of M with respect to e satisfy the following

$$\bar{k}_i(\bar{e}) = -(r^2/c^2)k_i(e) - (2r^2/c^2)(x \cdot e), \qquad i = 1, 2, \tag{7}$$

where $\bar{k}_i(\bar{e})$ are the corresponding principal curvatures on $\bar{M}$ and $\bar{e}=(2(x \cdot e)/r^2)x-e$. Hence we obtain

$$\begin{aligned}(\bar{k}_1(\bar{e}) + \bar{k}_2(e))^2 &- 4\bar{k}_1(\bar{e})\bar{k}_2(\bar{e}) \\ &= (r^4/c^4)\{(k_1(e) + k_2(e))^2 - 4k_1(e)k_2(e)\}.\end{aligned} \tag{8}$$

By taking averages of both sides of (8) over the spheres of unit normal vectors of $\bar{M}$ and M at the corresponding points, we obtain

$$\bar{H}^2 - \bar{K} = (r^4/c^4)(H^2 - K), \tag{9}$$

where $\bar{H}$ and $\bar{K}$ are the mean curvature and the Gauss curvature of $\bar{M}$. Hence, from (4) and (9), we obtain the Theorem.

REMARK 1. If M is an orientable closed surface in E^m, then, by combining the Theorem and the well-known Gauss-Bonnet formula, we see that the integral $\int_M H^2\, dV$ is a global conformal invariant. If the codimension is one, this invariant was observed by White [3].

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