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ON SPECIAL LINEAR CHARACTERS OF FREE GROUPS OF RANK $n \geqq 4$

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ABSTRACT. Let F_n be a free group of rank n. In a recent paper R. Horowitz has shown that for $n \leqq 3$ the ideal I_n in the ring of special linear characters of F_n consisting of those polynomials in the characters which vanish for all representations of F_n by subgroups of $SL(2, C)$ is principal. In this paper the case $n=4$ is investigated; it is shown that for $n>3$, I_n is not principal.

1. **Introduction.** The theory of two dimensional special linear characters was investigated extensively in the late nineteenth century by Fricke. More recently R. Horowitz [2] has studied the ring of special linear characters of free groups F_n; in particular, for $n=1,2$ he has shown that these rings are isomorphic respectively to the rings $\boldsymbol{Z}[x]$ and $\boldsymbol{Z}[x, y, z]$ of polynomials in one and three indeterminants with integer coefficients, and that the ring of characters of F_3 is isomorphic to the quotient ring of $\boldsymbol{Z}[x_1, x_2, \cdots, x_7]$ modulo a principal ideal consisting of those polynomials which vanish on the character manifold. The first theorem of this paper shows that the number of algebraic relations between the characters of F_n increases considerably with n; Theorem 2 gives some restriction on these relations for $n=4$.

2. **Preliminaries; statement of theorems.** Given a group G, denote by (G, K) the set of all representations $\rho: G \to SL(2, K)$, where K is either the field $\boldsymbol{R}$ or $\boldsymbol{C}$ of real or complex numbers, and where $SL(2, K)$ is the group of all 2×2 matrices with entries in K and with determinant one. For $u \in G$ the character σ_u of u is defined by $\sigma_u: (G, K) \to K$, with $\rho\sigma_u = \sigma(u\rho)$ for all $\rho \in (G, K)$, where $\sigma(u\rho)$ denotes the trace of $u\rho$ .The set $\{\sigma_u | u \in G\}$ generates a subring $J_{G,K}$ of the ring of all functions from (G, K) into K with the usual addition and multiplication.

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The following formulae for $u, v, w \in G$ [1] will be needed:

$$(2.1)\qquad \sigma_u = \sigma_{u^{-1}},$$

$$(2.2)\qquad \sigma_{uv} = \sigma_u \cdot \sigma_v - \sigma_{uv^{-1}},$$

$$(2.3)\qquad \sigma_{uvw} = \sigma_u \cdot \sigma_{vw} + \sigma_v \cdot \sigma_{uw} + \sigma_w \cdot \sigma_{uv} - \sigma_u \cdot \sigma_v \cdot \sigma_w - \sigma_{uwv}.$$

Using (2.1)–(2.3) Horowitz has proved the following theorem [2]:

Let F_n denote the free group on n free generators $a_k, k=1, \cdots, n$. For all $u \in F_n$, σ_u can be expressed as a polynomial with integer coefficients in the 2^n-1 characters

$$(2.4)\qquad \sigma_a, \qquad a = a_{i_1}a_{i_2} \cdots a_{i_r},\ 1 \leqq r \leqq n,\ 1 \leqq i_1 < \cdots < i_r \leqq n.$$

As an immediate consequence we have $J_{F_n,K} \cong Z_n/I_{n,K}$, where Z_n is the ring of polynomials with integer coefficients in the 2^{n-1} indeterminants $i_1i_2 \cdots i_r$, $1 \leqq r \leqq n$, $1 \leqq i_1 < \cdots < i_r \leqq n$, and $I_{n,K}$ is the ideal consisting of those polynomials in Z_n whose images in $J_{n,K}$ under the canonical homomorphism vanish for all $\rho \in (F_n, K)$. It is easily checked that $I_{n,C} \subseteq I_{n,R}$, and $m \leqq n$ implies $I_{m,K} \subseteq I_{n,K}$.

Each element φ of the group A_n of automorphisms of F_n induces a map $\sigma_a \to \sigma_{a\varphi}$ of the characters σ_a of (2.4), where according to Horowitz's theorem, $\sigma_{a\varphi}$ is a polynomial $p(\sigma_a)$ in the σ_a. The corresponding assignment $i_1i_2 \cdots i_r \to p(i_1i_2 \cdots i_r)$ induces an automorphism of Z_n, called the automorphism induced by φ. Clearly $I_{n,K}$ is invariant under the group $\mathscr{A}_n$ of automorphisms of Z_n induced by A_n. A set of generators for A_n is given by A, B, C, D, where the image of a generator a_k of F_n under each of these automorphisms is given by Table 1 [3, p. 164]:

TABLE 1

	a_1	a_2	a_3	$\cdots$	a_{n-1}	a_n
A	a_2	a_3	a_4	$\cdots$	a_n	a_1
B	a_2	a_1	a_3	$\cdots$	a_{n-1}	a_n
C	a_1^{-1}	a_2	a_3	$\cdots$	a_{n-1}	a_n
D	a_1a_2	a_2	a_3	$\cdots$	a_{n-1}	a_n

Let $\alpha, \beta, \gamma, \delta$ denote the automorphisms of $\mathscr{A}_n$ induced by A, B, C, D respectively. Finally, if $w = a_{j_1}^{\pm 1}a_{j_2}^{\pm 1} \cdots a_{j_k}^{\pm 1}$ is any nonempty word in the free generators $a_1, \cdots, a_n$ of F_n, denote σ_w by $\sigma_{\pm j_1 \pm j_2 \cdots \pm j_k}$. If $1 \leqq i < j < k \leqq n$ by (2.3) $\sigma_{ikj} = \sigma_i \cdot \sigma_{jk} + \sigma_j \cdot \sigma_{ik} + \sigma_k \cdot \sigma_{ij} - \sigma_i \cdot \sigma_j \cdot \sigma_k - \sigma_{ijk}$; let the symbol ikj represent the polynomial $i \cdot jk + j \cdot ik + k \cdot ij - i \cdot j \cdot k - ijk$ in

TABLE 2

	α	β	γ	δ
p_1	$p_1\alpha$	$*$	$*$	$*$
$p_1\alpha$	$p_1\alpha^2$	$p_1\alpha^2$	$*$	$*$
$p_1\alpha^2$	$p_1\alpha^3$	$p_1\alpha$	$*$	p_5
$p_1\alpha^3$	p_1	$*$	$*$	$*$
p_2	$p_2\alpha$	$p_2\alpha - 2\cdot p_6$	$p_2 + p_6$	$*$
$p_2\alpha$	$p_2\alpha^2$	$p_2 - 1\cdot p_6$	$-*$	$2\cdot p_2\alpha - p_6$
$p_2\alpha^2$	$p_2\alpha^3$	$-*$	$-*$	$p_3\beta + 2\cdot p_2\alpha^2$
$p_2\alpha^3$	p_2	$-*$	$-*$	$p_3\alpha$
p_3	$p_3\alpha$	$p_3\beta$	$p_3 - 1\cdot p_2\alpha^3$	$*$
$p_3\alpha$	p_3	$p_3\beta\alpha^2 - 1\cdot p_2\alpha^3 + 4\cdot p_2$	$-*$	$2\cdot p_3\alpha - p_2\alpha^3$
$p_3\beta$	$p_3\beta\alpha$	p_3	$-*$	$-p_2$
$p_3\beta\alpha$	$p_3\beta\alpha^2$	$p_3\beta\alpha + 3\cdot p_2\alpha^3$ $- 4\cdot p_2\alpha^2$	$-*$	$-p_4\alpha$
$p_3\beta\alpha^2$	$p_3\beta\alpha^3$	$p_3\alpha - 2\cdot p_2\alpha^3$ $- 4\cdot p_2\alpha + 2\cdot 4\cdot p_6$	$p_3\beta\alpha^2 + 1\cdot(4\cdot p_6 - p_2\alpha^3)$	$*$
$p_3\beta\alpha^3$	$p_3\beta$	$p_3\beta\alpha^3 - 2\cdot p_2 + 1\cdot p_2\alpha$	$p_3\beta\alpha^3 + 1\cdot p_2\alpha$	$*$
p_4	$p_4\alpha$	q_1	$p_4 + 1\cdot p_3\beta$	$*$
$p_4\alpha$	$p_4\alpha^2$	q_2	$-(p_4 + 4\cdot p_1)\alpha$	$2\cdot p_4\alpha + p_3\beta\alpha$
$p_4\alpha^2$	$p_4\alpha^3$	q_3	$p_4\alpha^2 + 1\cdot p_3\beta\alpha\beta$	$-(2\cdot p_5 + p_4\beta\alpha^2)$
$p_4\alpha^3$	p_4	q_4	$p_4\alpha^3 + 1\cdot p_3\alpha$	$p_4\alpha^3 - 12\cdot p_2\alpha^3$ $+ (2\cdot 12 - 1)p_3\alpha$
p_6	$*$	$-*$	$-*$	$p_2\alpha$

$\boldsymbol{Z}_n$. We can now state:

THEOREM 1. *In $\boldsymbol{Z}_4$, the set of polynomials*

$$\begin{aligned}
p_1 &= -123 \cdot 132 + 1^2 + 2^2 + 3^2 + (12)^2 + (13)^2 + (23)^2 \\
&\quad - 1 \cdot 2 \cdot 12 - 1 \cdot 3 \cdot 13 - 2 \cdot 3 \cdot 23 + 12 \cdot 13 \cdot 23 - \mathbf{4}, \\
p_2 &= 1 \cdot 1234 + 243 - 234 - 12 \cdot 134 + 13 \cdot 124 - 14 \cdot 123 \\
&\quad - 1 \cdot 3 \cdot 124 + 3 \cdot 12 \cdot 14, \\
p_3 &= 13 \cdot 1234 - 123 \cdot 134 - 1 \cdot 124 + 12 \cdot 14 - 2 \cdot 4 + \mathbf{2}(24) \\
&\quad - 3 \cdot 234 + 23 \cdot 34, \\
p_4 &= 132 \cdot 1234 - 12 \cdot 23 \cdot 134 - 23 \cdot 234 + 1 \cdot 3 \cdot 134 - 13 \cdot 134 \\
&\quad - 12 \cdot 124 + 2 \cdot 23 \cdot 34 + 2 \cdot 12 \cdot 14 + 2 \cdot 24 - 1 \cdot 14 \\
&\quad - 3 \cdot 34 + \mathbf{2}(4), \\
p_5 &= (1234)^2 - 1234(4 \cdot 123 + 3 \cdot 124 + 12 \cdot 34 - 3 \cdot 4 \cdot 12) \\
&\quad + (123)^2 + (124)^2 + (12)^2 + (34)^2 + 3^2 + 4^2 - 4 \cdot 12 \cdot 124 \\
&\quad - 3 \cdot 12 \cdot 123 - 3 \cdot 4 \cdot 34 + 34 \cdot 123 \cdot 124 - \mathbf{4}, \\
p_6 &= \mathbf{2}(1234) + (1 \cdot 2 - 12)34 + (1 \cdot 4 - 14)23 + (13 - 1 \cdot 3)24 \\
&\quad - 1 \cdot 234 - 2 \cdot 134 - 4 \cdot 123 + 3 \cdot 142
\end{aligned}$$

together with their images

$$\{p_j\alpha^i \mid i = 1, 2, 3, j = 1, 2, 4\} \cup \{p_3\alpha\} \cup \{p_3\beta\alpha^i \mid i = 0, 1, 2, 3\}$$

generate an ideal I which is (i) *invariant under $\mathscr{A}_4$*; (ii) *contained in $I_{4,K}$.*

Theorem 1 contrasts sharply with Horowitz's results that $I_{1,K}=I_{2,K}=\{0\}$, and $I_{3,K}$ is the principal ideal generated by p_1. Since $I_{m,K}\subseteq I_{n,K}$ for $m\leqq n$, the above result indicates the complexity of the ideals $I_{n,K}$ for $n\geqq 4$. On the other hand, the relations in $J_{F_4,K}$ are restricted by

THEOREM 2. *Let $p\in I_{4,R}$ be a polynomial of degree* 0 *in the indeterminants $i_1 i_2 \cdots i_r$ of length $r>2$ and of nonzero degree in at most nine of the ten indeterminants of length $r\leqq 2$. Then $p\equiv 0$.*

3. **Proof of Theorem 1.** Table 2 showing the images of the generating polynomials for I under the automorphisms α, β, δ, γ can be verified by direct calculation using (2.1)–(2.3) and the defining relations for A_4[3, p. 164]. The symbols $\pm^*$ indicate that a given polynomial is either fixed or sent to its negative by a given automorphism. The images of p_5 and the polynomials q_i ($i=1, 2, 3, 4$) of Table 2 because of their length, are written separately.

$$\begin{aligned}
p_5\alpha &= p_5 + 1\cdot p_4\alpha - 3\cdot p_4\alpha^3 - 34\cdot p_3\beta\alpha^3 - 1\cdot 34 p_2\alpha \\
&\quad + 14\cdot(p_3\beta + 2\cdot p_2\alpha^2), \\
p_5\beta &= p_5 + [1\cdot(1\cdot p_1\alpha + p_4\alpha + p_4\beta\alpha) + 4\cdot(4\cdot p_1 + p_4 + p_4\alpha^3\beta\alpha)]\alpha \\
&\quad + 1\cdot 2\cdot(p_3\beta\alpha + p_3\beta\alpha\beta + 34\cdot p_6), \\
p_5\gamma &= p_5 + 1\cdot[(4\cdot p_1 + p_4 + p_4\alpha^3\beta\alpha)\alpha], \\
p_5\delta &= 2^2\cdot p_5 + (4\cdot p_4\beta + 4\cdot p_4 + p_1)\alpha^2, \\
q_1 &= p_4 + (1\cdot 2 - 12)p_2\alpha^2 - 2p_3 + 1\cdot p_3\beta, \\
q_2 &= 1\cdot 4\cdot p_2\alpha^2 + 3\cdot p_3\alpha\beta \\
&\quad - (p_4\beta\alpha^2 + 2\cdot p_1\alpha^2 + 4\cdot p_3 + 1\cdot p_3\beta\alpha + 1\cdot 34\cdot p_6), \\
q_3 &= [23\cdot p_2 + (1\cdot 2 - 12)p_2\alpha^2 - p_4\beta - 4\cdot p_1 - 2\cdot p_3]\alpha - 2\cdot 34\cdot p_6, \\
q_4 &= p_4\alpha^3 + 2\cdot p_3\beta\alpha^2 + 2\cdot 4\cdot p_2 - 1\cdot p_3\alpha - 12\cdot p_2\alpha^3.
\end{aligned}$$

It is evident from the table that I is invariant under $\mathscr{A}_4$. In order to show $I\subseteq I_{4,K}$, because of the relations $p_2=-p_3\beta\delta$, $p_4=p_3\beta\alpha\delta\alpha^{-1}$, $p_5=-(p_4\alpha^2\delta+p_4\beta\alpha^2)$, $p_1=p_5(\alpha_1^2\delta)^{-1}$, $p_6=p_2\alpha\delta^{-1}$ and the fact that $I_{4,K}$ is invariant under $\mathscr{A}_4$, one need only show $p_3\in I_{4,K}$. For this purpose let $u=a_1a_2a_3$, $v=a_1a_3a_4$. By (2.2) we have

$$0 = -\sigma_{123}\cdot\sigma_{134} + \sigma_{123134} + \sigma_{23-3-4}. \tag{3.1}$$

Rewriting σ_{23-3-4} by (2.1) and (2.2) one obtains

$$\sigma_{23-3-4} = \sigma_{23}\sigma_{34} - \sigma_3\cdot\sigma_{234} + \sigma_{24}. \tag{3.2}$$

Expansion of σ_{123134} via (2.3) with $u=a_1a_2$, $v=a_3a_1a_3$, $w=a_4$ using (2.1), (2.2) yields

$$\sigma_{123134} = \sigma_{13}\cdot\sigma_{1234} - \sigma_1\cdot\sigma_{124} + \sigma_{12}\cdot\sigma_{14} - \sigma_2\cdot\sigma_4 + \sigma_{24}. \tag{3.3}$$

Substitution of (3.2), (3.3) into (3.1) yields the desired result.

4. **Proof of Theorem 2.** We may assume without loss of generality that p has degree 0 in either of the indeterminants 4 or 34. Indeed, if p has degree 0 in one of the remaining eight indeterminants of length $k\leqq 2$, p may be replaced by $p\alpha^\lambda\beta^\varepsilon$ for suitable $\lambda=0, 1, 2, 3$, $\varepsilon=0, 1$. Both cases are treated simultaneously.

Define a representation $\rho: F_4\to SL(2, \boldsymbol{R})$ as follows:

$$a_1\rho = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{bmatrix},\quad a_2\rho = \begin{bmatrix}\lambda_2 & \lambda_2\lambda_3 - 1\\ 1 & \lambda_3\end{bmatrix},\quad a_3\rho = \begin{bmatrix}\lambda_4 & \lambda_5\\ \lambda_6 & \dfrac{\lambda_5\lambda_6+1}{\lambda_4}\end{bmatrix},$$

$$a_4\rho = \begin{bmatrix}\lambda_7 & \lambda_8\\ \lambda_9 & \dfrac{\lambda_8\lambda_9+1}{\lambda_7}\end{bmatrix},\qquad \lambda = (\lambda_1, \cdots, \lambda_9)\in \boldsymbol{R}_9,\ \lambda_1, \lambda_4, \lambda_7 \text{ nonzero}.$$

The two systems of functional equations $\lambda(\rho\sigma_i)=\xi_i$, $i=1, 2, 3, 12, 13, 14, 23, 24, k$, $k=4$ and $k=34$, define transformations F_1 and F_2 respectively from a subregion of $\boldsymbol{R}_9$ into $\boldsymbol{R}_9$. Let $\bar{\lambda} \in \boldsymbol{R}_9$ have coordinates $\lambda_1=2$, $\lambda_i=1$, $1<i\leqq 9$. The Jacobeans of both F_1 and F_2 evaluated at $\bar{\lambda}$ are nonzero. It follows from the implicit function theorem that there exist neighborhoods U_i of $\bar{\lambda}F_i$ and transformations $G_i: U_i \rightarrow \boldsymbol{R}_9$ $(i=1, 2)$ with the property that $\xi(G_i \circ F_i)=\xi$ for all $\xi \in U_i$. Accordingly, for all $\xi \in U_i$ there exist representations $\rho_i \in (F_4, \boldsymbol{R})$ $(i=1, 2)$ such that $\xi=(\rho_1\sigma_1, \cdots, \rho_1\sigma_{24}, \rho_1\sigma_4)$, $i=1$; $\xi=(\rho_2\sigma_1, \cdots, \rho_2\sigma_{24}, \rho_2\sigma_{34})$, $i=2$.

Since $p \in I_{4,\boldsymbol{R}}$ we have $p(\xi)=0$ for all $\xi \in U_i$, which implies $p\equiv 0$.

References

1. R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Functionen*. Vol. 1, Teubner, Leipzig, 1897.

2. R. Horowitz, *Characters of free groups represented in the two-dimensional special linear group*, Comm. Pure Appl. Math. **25** (1972), 635-650.

3. W. Magnus, A. Karass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure and Appl. Math., vol. 13, Interscience, New York, 1966. MR **34** #7617.

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