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BOUNDED HOLOMORPHIC FUNCTIONS IN SIEGEL DOMAINS

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ABSTRACT. A Siegel domain D of the second kind (not necessarily affine homogeneous) is shown to be complete with respect to the Carathéodory distance. Thus D is convex with respect to the bounded holomorphic functions, hence is a domain of holomorphy. A Phragmén-Lindelöf theorem for D is also given. That is, if a holomorphic function f in D is continuous in $\bar{D}$, bounded on the distinguished boundary S of D and not of exponential growth, then f is bounded in $\bar{D}$.

1. **Introduction.** The purpose of this paper is to prove two theorems concerning bounded holomorphic functions in Siegel domains of the second kind.

THEOREM 1. *A Siegel domain D of the second kind (not necessarily affine homogeneous) is complete with respect to the Carathéodory distance on D.*

COROLLARY 1. *A Siegel domain D of the second kind (not necessarily affine homogeneous) is convex with respect to the bounded holomorphic functions, hence is a domain of holomorphy.*

COROLLARY 2. *A bounded homogeneous domain in C^n is convex with respect to the bounded holomorphic functions, hence is a domain of holomorphy.*

THEOREM 2 (PHRAGMÉN-LINDELÖF). *Let $f(z, u)$ be a holomorphic function in a Siegel domain D of the second kind and continuous in $\bar{D}$. Suppose that $|f(z, u)| \leqq M$ on the distinguished boundary S of D and $f(z, u)$ is not of exponential growth. Then $|f(z, u)| \leqq M$ in $\bar{D}$.*

2. **Siegel domains of the second kind.** Let Ω be a regular cone in $\boldsymbol{R}^n$, i.e. a nonempty convex open set such that $0 \neq x \in \bar{\Omega}$ and $\lambda > 0$ imply $\lambda x \in \bar{\Omega}$, $-x \notin \bar{\Omega}'$. $\bar{\Omega}'$ denotes the dual cone, i.e. the set of all real linear

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functionals α on $\boldsymbol{R}^n$ such that $\langle x, \alpha\rangle > 0$ for all $0 \neq x \in \bar{\Omega}$. Let $\Phi : \boldsymbol{C}^m \times \boldsymbol{C}^m \to \boldsymbol{C}^n$ be a hermitian bilinear form with respect to $\boldsymbol{R}^n$ such that $\Phi(u, u) \in \bar{\Omega}$ for all $u \in \boldsymbol{C}^m$ and let $D \subset \boldsymbol{C}^{n+m}$ be the domain

$$D = \{(z, u) \in \boldsymbol{C}^{n+m} \mid \operatorname{Im} z - \Phi(u, u) \in \Omega\}.$$

D is called a Siegel domain of the second kind. If $m=0$, then D is called a Siegel domain of the first kind or a radial tubular domain over Ω. The distinguished boundary S is the set

$$\{(z, u) \in \boldsymbol{C}^{n+m} \mid \operatorname{Im} z = \Phi(u, u)\}.$$

It is known that if F is a bounded continuous function in $\bar{D}$ and holomorphic in D, then $\sup_D |f(z, u)| = \sup_S |f(z, u)|$. A theorem of Gindikin, Pyatetskii-Shapiro and Vinberg **[6]** says that every bounded homogeneous domain in $\boldsymbol{C}^n$ is biholomorphic to an affine homogeneous Siegel domain of the second kind.

3. **Bounded holomorphic convexity of Siegel domains.** Let d_M and c_M denote the Kobayashi and Carathéodory pseudodistances respectively on a complex manifold M. (For definitions, see **[4]**.) A complex manifold is said to be convex with respect to the bounded holomorphic functions (convex with respect to $B(M)$) if

$$\hat{K}_B = \{x \in M \mid |f(x)| \leqq \|f\|_K, \text{ for all } f \in B(M)\}$$

($B(M)$=the algebra of bounded holomorphic functions on M) is compact provided K is a compact subset of M. A theorem of S. Kobayashi **[4]** says that if M is complete with respect to the Carathéodory distance, then M is convex with respect to $B(M)$. Here we shall prove that a Siegel domain D of the second kind is complete with respect to the Carathéodory distance on D. The proof is quite trivially implied by Kobayashi's book **[4]**. However this fact is still worthwhile to be pointed out. For instance, the well-known theorem that a bounded homogeneous domain is a domain of holomorphy is a corollary of this fact. Another consequence is that a radial tubular domain is a domain of holomorphy **[1]**. Moreover, convexity with respect to $B(M)$ is much stronger than holomorphy. The domain $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1| < |z_2| < 1\}$ is a domain of holomorphy but is not convex with respect to $B(M)$. In **[3]**, D. S. Kim has shown that convexity with respect to $B(M)$ implies bounded holomorphy. A domain of bounded holomorphy is a maximal domain for which every bounded holomorphic function has a bounded analytic continuation. The punctured disk $0 < |z| < 1$ is a domain of holomorphy but is not a domain of bounded holomorphy. The domain $H \times \boldsymbol{C}^{n-1} \subset \boldsymbol{C}^n$, where $H = \{z \in \boldsymbol{C}^1 \mid \operatorname{Re} z > 0\}$, is a domain of bounded holomorphy but is not convex with respect to $B(M)$.

PROOF OF THEOREM 1. According to [4, p. 64], a Siegel domain of the second kind can be written as the intersection of (possibly uncountably many) domains each of which is biholomorphic to a product of balls. A product of balls is known to be complete hyperbolic with respect to the Kobayashi distance. Since for a bounded symmetric domain the Kobayashi and Carathéodory distances coincide, a product of balls is complete with respect to the Carathéodory distance. Let M and M_i $(i \in I)$ be complex submanifolds of a complex manifold N such that $M=\bigcap_{i\in I} M_i$. If each M_i is complete with respect to its Carathéodory distance, so is M. (See [4].) Consequently, a Siegel domain of the second kind is complete with respect to the Carathéodory distance.

4. **A Phragmén-Lindelöf theorem.** Let f be a holomorphic function in a domain D of the complex plane C^1 between two straight lines making an angle at the origin and continuous in $\bar{D}$. Suppose that $|f(z)| \leqq M$ on the lines. The Phragmén-Lindelöf theorem says that either $|f(z)| \leqq M$ in $\bar{D}$ or f is of exponential growth [8]. We say that a function f is not of exponential growth or

$$|f(z, u)| = o(\exp(|z_1|^\gamma + \cdots + |z_n|^\gamma + \langle \Phi(u, u), \alpha\rangle)),$$

if

$$|f(z, u)|\, |\exp(-(|z_1|^\gamma + \cdots + |z_n|^\gamma + \langle \Phi(u, u), \alpha\rangle))| \to 0$$

whenever $\sum_k |z_k| + \sum_j |u_j| \to \infty$, for $(z, u) \in D$, a fixed $\alpha \in \bar{\Omega}'$ and a fixed number γ $(0<\gamma<1)$.

PROOF OF THEOREM 2. Without loss of generality, we may assume that our domain D is $\{(z, u) | \operatorname{Re} z - \Phi(u, u) \in \Omega\}$ by a rotation. Consider the function $F(z, u) = \exp(-\varepsilon(z_1^\gamma + \cdots + z_n^\gamma)) \exp(-\varepsilon\langle z, \alpha\rangle) f(z, u)$, where ε is a positive real number. Then F is holomorphic in D, because the first factor is holomorphic in a larger domain containing D. This larger domain (see [2], [6]) is the product of

$$\operatorname{Re} z_1 - (|u_1'|^2 + \cdots + |u_{m_1}'|^2) > 0,$$
$$\cdots$$
$$\operatorname{Re} z_n - (|u_{m_{n-1}+1}'|^2 + \cdots + |u_m'|^2) > 0.$$

Let $z_k = r_k e^{i\theta_k}$, then $-(\pi/2) \leqq \theta_k \leqq (\pi/2)$ in $\bar{D}$. On S, we have

$$\begin{aligned} |F(z, u)| &= \exp(-\varepsilon(r_1^\gamma \cos \gamma\theta_1 + \cdots + r_n^\gamma \cos \gamma\theta_n)) \\ &\quad \cdot \exp(-\varepsilon\langle \operatorname{Re} z, \alpha\rangle) \cdot |f(z, u)| \\ &= \exp(-\varepsilon(r_1^\gamma \cos \gamma\theta_1 + \cdots + r_n^\gamma \cos \gamma\theta_n)) \\ &\quad \cdot \exp(-\varepsilon\langle \Phi(u, u), \alpha\rangle) \cdot |f(z, u)| \\ &\leqq |f(z, u)| \leqq M. \end{aligned}$$

Moreover, for $(z, u) \in D$,

$$|F(z, u)| \leqq \exp(-\varepsilon(r_1^\gamma \cos \gamma\theta_1 + \cdots + r_n^\gamma \cos \gamma\theta_n)) \cdot \exp(-\varepsilon\langle\Phi(u, u), \alpha\rangle) \cdot |f(z, u)| \to 0,$$

whenever $\sum_k |z_k| + \sum_j |u_j| \to \infty$ by the assumption. Consequently F is bounded and continuous in $\bar{D}$ and holomorphic in D. Since $|F(z,u)| \leqq M$ on S, $|F(z, u)| \leqq M$ in $\bar{D}$. Let $\varepsilon \to 0$. We obtain finally $|f(z, u)| \leqq M$ for all $(z, u) \in \bar{D}$.

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