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CERTAIN SUBSETS OF PRODUCTS OF θ-REFINABLE SPACES ARE REALCOMPACT

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ABSTRACT. It is shown that the normal T_1-space X is realcompact if and only if (a) each discrete subset of X is realcompact and (b) X can be embedded as a closed subset in the product of a collection of regular θ-refinable spaces.

We will say that a space X has property (*) if it is true that each discrete subset of X is realcompact; i.e., the cardinality of each discrete subset of X is nonmeasurable. In [5], the author has shown that a normal T_1-space X is realcompact if and only if X has property (*) and X can be embedded as a closed subspace in the product of a collection of subparacompact spaces and metacompact spaces. S. Mrówka suggested to the author that there should be a nontrivial class of spaces $\mathscr{P}$ containing the class of subparacompact spaces and the class of metacompact spaces so that a normal space X is realcompact if and only if X has property (*) and X can be embedded as a closed subspace in a product of members of $\mathscr{P}$. It is the purpose of this paper to show that the class of *θ-refinable* spaces, introduced by Worrell and Wicke in [4], is such a class.

Recall that a space X is *θ-refinable* if it is true that if $\mathscr{V}$ is an open cover of X then there is a sequence $\mathscr{V}_1, \mathscr{V}_2, \cdots$ of open covers of X that refine $\mathscr{V}$ such that if $x \in X$, then there is an integer i such that only finitely many members of $\mathscr{V}_i$ contain x. Clearly, any metacompact space is θ-refinable. It is shown in [1] that any subparacompact space is θ-refinable.

Our notation will follow that of [2].

LEMMA 1 [5]. *Suppose that X is a T_1-space and $\mathscr{E}$ is a class of T_3-spaces such that the topology on X is the weak topology induced by $C(X, \mathscr{E})$. Then X can be embedded as a closed subspace in the product of a collection of members of $\mathscr{E}$ if and only if it is true that if $\mathscr{F}$ is a free ultrafilter of closed subsets of X, then there are a member f of $C(X, \mathscr{E})$ and an open cover $\mathscr{U}$ of range (f) such that $\{f^{-1}(U): U \in \mathscr{U}\}$ refines $\{(X-F): F \in \mathscr{F}\}$.*

LEMMA 2 (THEOREM 18, [3]). *If $\mathscr{U}$ is an open cover of the space X, then*

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there is a discrete subspace H of X such that

(i) $\{\mathrm{st}(x, U): x \in H\}$ *covers* X, *and*

(ii) *no member of* $\mathcal{U}$ *contains two points of* H.

THEOREM. *The following conditions on a normal T_1-space X are equivalent*: (1) *X is realcompact.*

(2) *X has property* (*) *and X can be embedded as a closed subset in the product of a collection of regular θ-refinable spaces.*

(3) *X has property* (*) *and if $\mathcal{F}$ is a free ultrafilter of closed subsets of X, then there is a sequence $\mathcal{W}_1, \mathcal{W}_2, \cdots$ of open covers of X refining $\{X-F \mid F \in \mathcal{F}\}$ such that if $x \in X$ then there is an integer i such that only finitely many members of $\mathcal{W}_i$ contain x.*

PROOF. (1) *implies* (2). This is obvious since every closed subset of a realcompact space is realcompact and the real line is θ-refinable.

(2) *implies* (3). Let $\mathcal{F}$ be a free ultrafilter of closed subsets of X. According to Lemma 1, there are a θ-refinable space Y, an open cover $\mathcal{V}$ of Y, and a continuous function f taking X into Y such that $f^{-1}(\mathcal{V})=\{f^{-1}(V): V \in \mathcal{V}\}$ refines $\{X-F \mid F \in \mathcal{F}\}$. Since Y is θ-refinable, there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of open covers of Y refining $\mathcal{V}$ such that if $y \in Y$, there is an integer i such that only finitely many members of $\mathcal{V}_i$ contain x. Clearly, if for each i, $\mathcal{W}_i=f^{-1}(\mathcal{V}_i)$, the sequence $\mathcal{W}_1, \mathcal{W}_2, \cdots$ satisfies condition (3) of our theorem.

(3) *implies* (1). Suppose that X satisfies condition (3) but X is not realcompact. Let $\mathcal{Z}$ be a free Z-ultrafilter in X with the countable intersection property. Let $\mathcal{F}$ be the ultrafilter of closed subsets of X that contains $\mathcal{Z}$ ($\mathcal{F}$ is uniquely determined by $\mathcal{Z}$ since X is normal). Let $\mathcal{W}_1, \mathcal{W}_2, \cdots$ be a sequence of open covers of X refining $\{X-F: F \in \mathcal{F}\}$ such that if $x \in X$ then there is an i such that only finitely many members of $\mathcal{W}_i$ contain x. For each pair (i,j) of positive integers, let $H(i,j)=\{x \in X \mid x$ is contained in at most j members of $\mathcal{W}_i\}$. It is easy to see that each $H(i,j)$ is closed. Let $\mathcal{H}$ denote collection of all $H(i,j)$'s. Let $\mathcal{H}_1=\mathcal{H}-\mathcal{F}$ and $\mathcal{H}_2=\mathcal{H}-\mathcal{H}_1$. For each H in $\mathcal{H}_1$, let $F(H)$ denote a member of $\mathcal{F}$ that does not intersect H. Since X is normal, there is a zero-set $Z(H)$ containing $F(H)$ that does not intersect H. For each H in $\mathcal{H}_1$, $Z(H)$ is in $\mathcal{Z}$. It must be the case that $\mathcal{H}_2$ is not empty; otherwise, $\{Z(H): H \in \mathcal{H}_1\}$ would be a countable subcollection of $\mathcal{Z}$ with no common part which would be a contradiction.

For each $H=H(i,j)$ in $\mathcal{H}_2$, there is, by Lemma 2, a discrete subset $K(H)$ of H such that no member of $\mathcal{W}_i$ contains two members of $K(H)$ and $\{\mathrm{st}(x, W_i) \mid x \in K(H)\}$ covers H. Note that $K(H)$ is infinite for otherwise, the collection $\{W \in \mathcal{W}_i: W \cap K(H) \neq \varnothing\}$ would be finite and $\bigcap \{X-W: W \in \mathcal{W}_i, W \cap K(H) \neq \varnothing\}$ would be a member of $\mathcal{F}$ that would not intersect H and this would contradict the assumption that $H \in \mathscr{H}_2$. Let $\mathscr{W}'_i = \{W \in \mathscr{W}_i : W \cap K(H) \neq \varnothing\}$. Since $K(H)$ is infinite and each point of H is contained in only finitely many members of $\mathscr{W}_i$, it must be true that the cardinality of $K(H)$ is the same as the cardinality of $\mathscr{W}'_i$. Let φ be a one-to-one function from $K(H)$ onto $\mathscr{W}'_i$. For each F in $\mathscr{F}$, let $M(F) = \{x \in K(H) : \varphi(x) \cap (F \cap H) \neq \varnothing\}$. Clearly, $\{M(F) : F \in \mathscr{F}\}$ has the finite intersection property; and so, there is an ultrafilter $\mathscr{M}$ of subsets of $K(H)$ that contains $\{M(F) : F \in \mathscr{F}\}$. Since, for each $x \in K(H)$, it is true that $X - \varphi(x) \in \mathscr{F}$, it is true that $\mathscr{M}$ is a free ultrafilter of subsets of $K(H)$. Since $K(H)$ is a discrete subset of X, $K(H)$ is realcompact; and so, there is a countable subcollection $\{M_i\}$ of members of $\mathscr{M}$ with no common part.

Claim 1. If $M \in \mathscr{M}$, there is a member F of $\mathscr{F}$ that is a subset of $\bigcup_{x \in M} \varphi(x)$.

The argument for this is the same as the argument for Claim 1 in the proof of the theorem in [**5**].

Claim 2. $[\bigcap_{i=1}^{\infty} (\bigcup_{x \in M_i} (\varphi(x)))] \cap H = \varnothing$.

Again, the argument for this is the same as the argument for Claim 2 in the proof of the theorem in [**5**].

By Claim 1, for each integer n, there is a member F_n of $\mathscr{F}$ such that $F_n \subset \bigcup_{x \in M_n} (\varphi(x))$. Since X is normal, there is a zero-set Z_n such that $F_n \subset Z_n \subset \bigcup_{x \in M_n} (\varphi(x))$. It follows from Claim 2 that $\bigcap (Z_n \cap H) = \varnothing$. Thus, for each $H \in \mathscr{H}_2$ there is a countable subcollection $\mathscr{Z}(H)$ of $\mathscr{Z}$ such that $[\bigcap_{Z \in \mathscr{Z}(H)} (Z)] \cap H = \varnothing$. Thus, we have $\{Z(H) | H \in \mathscr{H}_1\} \cup (\bigcup_{H \in \mathscr{H}_2} \mathscr{Z}(H))$ is a countable subcollection of $\mathscr{Z}$ with no common part which contradicts the assumption that $\mathscr{Z}$ has the countable intersection property.

Note. In [**5**], the author asked if every normal metacompact space is topologically complete (in the sense of Dieudonné). R. Haydon offers an example of a normal metacompact space which is not complete in [**6**].

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