ANOSOV FLOWS AND EXPANSIVENESS

VICTOR NORTON AND THOMAS O'BRIEN¹

ABSTRACT. We prove that an Anosov diffeomorphism of a compact manifold is expansive. We also show that a continuous flow on an infinite compact metric space cannot be expansive. We define a corresponding expansive concept for flows, that of an unstable flow (the word unstable is used here in a Lyapunov sense). We then prove that Anosov flows of compact manifolds are unstable.

1. Introduction. Let X be a metric space with metric d and G a topological group. A transformation group (X, G, π) is called *expansive* if there is an $\varepsilon > 0$ (called an expansive constant) such that for any pair of distinct points x, y in X there is a $g \in G$ such that $d(xg, yg) > \varepsilon$. Here, as is customary, xg is used in place of $\pi(x, g)$.

If G is the integers then the transformation group is generated by a homeomorphism $f: X \rightarrow X$. In this case f is called an *expansive homeomorphism*. There are numerous examples of expansive homeomorphisms on compact spaces [2]. On the other hand if G is the reals R and X is an infinite compact space then (X, R, π) cannot be expansive (Proposition 3.1). When $G = R$, the transformation group will be called a *flow*. For flows, we will use $\gamma(x)$ to denote the orbit of the point x .

DEFINITION 1.1. A flow $\pi: X \times R \rightarrow X$ is called *unstable* if there is an $\varepsilon > 0$ such that for any x, y in X with $y \notin \gamma(x)$ there is a t (possibly negative) for which $d(xt, yt) > \varepsilon$. Thus, if π is unstable, no orbit can be both positively and negatively Lyapunov stable (here we assume X is self-dense; X will be a manifold for the remainder of the paper). In §2 we prove that an Anosov diffeomorphism on a compact manifold is expansive. This is indicated in [3, Proposition 8.7] but a proof has not been published. In §3 we prove that an Anosov flow on a compact manifold is unstable.

2. Anosov diffeomorphisms. A diffeomorphism f of a compact Riemannian manifold M is called an *Anosov diffeomorphism* if there is a splitting of the tangent bundle $TM = E^u + E^s$, invariant under the derivative Df ,

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and positive numbers a , b , and λ such that:

$$\begin{aligned} \|Df^n \cdot \xi^u\| &\geq ae^{\lambda n} \|\xi^u\|, & \|Df^{-n} \cdot \xi^u\| &\leq be^{-\lambda n} \|\xi^u\|, \\ \|Df^n \cdot \xi^s\| &\leq be^{-\lambda n} \|\xi^s\|, & \|Df^{-n} \cdot \xi^s\| &\geq ae^{\lambda n} \|\xi^s\|, \end{aligned}$$

for every $\xi = \xi^u + \xi^s$ in TM .

We will assume that

$$\begin{aligned} \|Df \cdot \xi^u\| &\geq 7 \|\xi^u\|, & \|Df^{-1} \cdot \xi^u\| &\leq \frac{1}{7} \|\xi^u\|, \\ \|Df \cdot \xi^s\| &\leq \frac{1}{7} \|\xi^s\|, & \|Df^{-1} \cdot \xi^s\| &\geq 7 \|\xi^s\|, \end{aligned}$$

for each ξ in TM . These inequalities hold for some iterate of f , and f is expansive if that iterate of f is expansive.

This last set of inequalities implies that, given any tangent vector ξ in TM , $\|Df \cdot \xi\| \geq 3 \|\xi\|$ if $\|\xi^u\| \geq \|\xi^s\|$ and $\|Df^{-1} \cdot \xi\| \geq 3 \|\xi\|$ if $\|\xi^s\| \geq \|\xi^u\|$. For instance when $\|\xi^u\| \geq \|\xi^s\|$ we have

$$\begin{aligned} \|Df \cdot \xi\| &= \|Df \cdot \xi^u + Df \cdot \xi^s\| \geq \|Df \cdot \xi^u\| - \|Df \cdot \xi^s\| \\ &\geq 7 \|\xi^u\| - \frac{1}{7} \|\xi^s\| \geq 6 \|\xi^u\| \geq 3 \|\xi\|. \end{aligned}$$

Let us denote the zero cross-section of TM by 0_* . For $r > 0$ and p in M , $B(r)_p$ is the open ball of radius r about 0 in TM_p . We let $B(r)$ denote the open tube of radius r about 0_* in TM , $B(r) = \bigcup B(r)_p$ ($p \in M$).

Let $\pi: TM \rightarrow M$ be the projection and $\exp$ the exponential map. The map $\Psi: \xi \rightarrow (\pi\xi, \exp \xi)$ restricts to a diffeomorphism of some open neighborhood of 0_* in TM onto an open neighborhood of the diagonal in $M \times M$. In order to simplify later measurements we lift f and f^{-1} to TM via Ψ , defining $F = \Psi^{-1} \cdot (f \times f) \cdot \Psi$, $G = \Psi^{-1} \cdot (f^{-1} \times f^{-1}) \cdot \Psi$. Then if r is taken sufficiently small, F and G restrict to diffeomorphisms of $B(r)$ onto open neighborhoods U and V of 0_* in TM . We assume further that $d(\pi\xi, \exp \xi) = \|\xi\|$ for any ξ in $B(r) \cup U \cup V$.

Let F_p and G_p denote the restrictions of F and G to the fibre $B(r)_p$. Then F_p and G_p are maps of Euclidean spaces. We have the relations:

$$DF_p(0) \cdot \xi = Df \cdot \xi, \quad DG_p(0) \cdot \xi = Df^{-1} \cdot \xi$$

for every p in M and ξ in TM_p . These equations are a consequence of the chain rule and the fact that $D(\exp_p)(0): TM_p \rightarrow TM_p$ is the identity map.

LEMMA 2.1. *Let $f: M \rightarrow M$ be an Anosov diffeomorphism of the compact Riemannian manifold M . There is a positive number ε such that either $d(f(p), f(q)) \geq 2d(p, q)$ or $d(f^{-1}(p), f^{-1}(q)) \geq 2d(p, q)$ whenever $d(p, q) < \varepsilon$.*

PROOF. The map $\xi \rightarrow \|DF_{\pi\xi}(\xi) - DF_{\pi\xi}(0)\|$ of $B(r)$ into R is continuous and by compactness we can choose a number ε_F less than or equal to r such that $\|DF_{\pi\xi}(\xi) - DF_{\pi\xi}(0)\| \leq 1$ for any ξ in $B(\varepsilon_F)$ (the norm of the linear transformation $A = DF_{\pi\xi}(\xi) - DF_{\pi\xi}(0)$ of $TM_{\pi\xi}$ is the usual sup norm, $\|A\| = \sup\{\|A \cdot \eta\| : \|\eta\| \leq 1\}$). Likewise choose ε_G less than or equal

to r such that $\|DG_{\pi\xi}(\xi) - DG_{\pi\xi}(0)\| \leq 1$ when ξ is in $B(\varepsilon_G)$. Then setting $\varepsilon = \min\{\varepsilon_F, \varepsilon_G\}$, we have the estimates [1, I, §4, Corollary 2],

$$\|F_p(\xi) - Df \cdot \xi\| \leq \|\xi\|, \quad \|G_p(\xi) - Df^{-1} \cdot \xi\| \leq \|\xi\|,$$

for every p in M and ξ in $B(\varepsilon)_p$.

Now suppose p and q are given with $d(p, q) < \varepsilon$. Take ξ in $B(\varepsilon)_p$ satisfying $\exp \xi = q$. If $\|\xi^u\| \geq \|\xi^s\|$,

$$\begin{aligned} d(f(p), f(q)) &= \|F_p(\xi)\| \geq \|Df \cdot \xi\| - \|\xi\| \\ &\geq 3\|\xi\| - \|\xi\| = 2\|\xi\| = 2d(p, q). \end{aligned}$$

On the other hand if $\|\xi^s\| \geq \|\xi^u\|$ we have $d(f^{-1}(p), f^{-1}(q)) \geq 2d(p, q)$ by an analogous computation.

Applying Lemma 2.1 repeatedly we have:

THEOREM 2.2. *An Anosov diffeomorphism of a compact manifold M is expansive (the ε of Lemma 2.1 is an expansive constant).*

3. Anosov flows. In this section M is again a compact Riemannian manifold and the notations are the same as in §2.

PROPOSITION 3.1. *A flow on an infinite compact space cannot be expansive.*

PROOF. Let X be infinite and compact and $\phi: X \times \mathbb{R} \rightarrow X$ a flow. Let ε be greater than 0, $I = [0, 1]$, and $\psi = \phi|_{X \times I}$. Then if ρ is the product metric on $X \times I$, there is a $\delta > 0$ such that $\rho((y, t_1), (z, t_2)) < \delta$ implies $d(yt_1, zt_2) < \varepsilon$. If ϕ is to be expansive there can be only finitely many rest points. Let x be a nonrest point and choose a sequence of positive numbers $\{t_n\}$ decreasing monotonically to 0 so that $x \neq xt_n$ for any n . If ε is an expansive constant for each n there is an s_n such that $d(xs_n, (xt_n)s_n) > \varepsilon$. However for n large enough $\rho((xs_n, 0), (xs_n, t_n)) < \delta$ and hence $d(xs_n, (xs_n)t_n) = d(xs_n, (xt_n)s_n) < \varepsilon$.

A differentiable flow $\phi: M \times \mathbb{R} \rightarrow M$ induced by a vector field $\mathcal{V}$ is an Anosov flow if

(i) ϕ has no rest points,

(ii) there is an invariant decomposition $TM = E^s + E^u + E^0$ satisfying the following: there exist positive constants a , b , and λ such that for any $\xi = \xi^s + \xi^u + \xi^0$ in TM ,

$$\|D\phi_t \cdot \xi^u\| \geq ae^{\lambda t} \|\xi^u\| \quad \text{for all } t \geq 0,$$

$$\|D\phi_t \cdot \xi^s\| \leq be^{-\lambda t} \|\xi^s\| \quad \text{for all } t \geq 0,$$

$$\|D\phi_t \cdot \xi^u\| \leq be^{-\lambda t} \|\xi^u\| \quad \text{for all } t \leq 0,$$

$$\|D\phi_t \cdot \xi^s\| \geq ae^{\lambda t} \|\xi^s\| \quad \text{for all } t \leq 0.$$

(iii) E^0 has dimension one and is tangent to the flow.

Note we can always choose a positive number T so that

$$(3.2) \quad \begin{aligned} \|D\phi_T \cdot \xi^u\| &> 7 \|\xi^u\|, & \|D\phi_{-T} \cdot \xi^u\| &< \frac{1}{7} \|\xi^u\|, \\ \|D\phi_T \cdot \xi^s\| &< \frac{1}{7} \|\xi^s\|, & \|D\phi_{-T} \cdot \xi^s\| &> 7 \|\xi^s\|. \end{aligned}$$

There is a local codimension one submanifold N_p at p whose tangent space at p is $E_p^s + E_p^u$. N_p is transverse to the trajectories of the flow and is the image under $\exp_p$ of a neighborhood of 0 in $E_p^s + E_p^u$.

LEMMA 3.3. *Let $\phi: M \times R \rightarrow M$ be an Anosov flow. There exist T and ε greater than 0 such that if p is in M and q is in N_p and $d(p, q) < \varepsilon$ then either $d(\phi_T(p), \phi_T(q)) \geq 2d(p, q)$ or $d(\phi_{-T}(p), \phi_{-T}(q)) \geq 2d(p, q)$.*

PROOF. Let T be chosen so that the inequalities, 3.2, are satisfied. Then the proof is essentially the same as that of Lemma 2.1 with ϕ_T replacing f and ϕ_{-T} replacing f^{-1} . In particular, the ε of this lemma, and of the following theorem, is small enough so that the exponential map is defined on the ε -tube about the zero section of the tangent bundle of M .

THEOREM 3.4. *An Anosov flow on a compact manifold M is unstable.*

PROOF. Let $\phi: M \times R \rightarrow M$ be an Anosov flow and let ε and T be as in the preceding lemma. Choose $\alpha > 0$ such that if $d(p, q) < \alpha$ then $d(\phi_T(p), \phi_T(q)) < \varepsilon/2$ and $d(\phi_{-T}(p), \phi_{-T}(q)) < \varepsilon/2$. Now choose $\eta > 0$ such that if $d(p, q) < \eta$ then there is a q_p in $\gamma(q)$ with q_p in N_p and $d(q, q_p) < \alpha$. The existence of η is guaranteed by the fact that trajectories are transverse to N_p . Let $\delta = \min(\eta, \varepsilon/2)$. Suppose p, q are in M , $q \notin \gamma(p)$ and $d(p, q) < \delta$. By the lemma either $d(\phi_T(p), \phi_T(q_p))$ or $d(\phi_{-T}(p), \phi_{-T}(q_p))$ is greater than $2d(p, q_p)$. Also $d(\phi_{\pm T}(q), \phi_{\pm T}(q_p)) < \varepsilon/2$. Thus for some integer k we must have either $d(\phi_{kT}(p), \phi_{kT}(q)) > \eta$ and we cannot guarantee the existence of q_p or else $d(\phi_{kT}(p), \phi_{kT}(q)) > \varepsilon/2$ and the exponential map is no longer defined. Thus δ is an instability constant for ϕ and the proof is completed.

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DEPARTMENT OF MATHEMATICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403