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ZERO SETS OF FUNCTIONS FROM NON-QUASI-ANALYTIC CLASSES

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ABSTRACT. Any closed subset E of the real numbers R is the zero set of some C^∞-function f. One can also specify the order $d(s)$ of the zero of f at each element s of the set S of isolated points of E. The present note improves this result by showing that each non-quasi-analytic class $C\{M_n\}$ contains such functions.

R. B. Hughes stated the foregoing interesting result in [**1**]. The purpose of this note is to present a short proof of Hughes' theorem based on properties of H. E. Bray's construction presented on pp. 79–84 of [**2**].

Suppose that $\{M_n\}_{n=1}^\infty$ is a sequence of positive numbers; set $\mu_1=M_1^{-1}$ and $\mu_k=(M_{k-1}/M_k)$, $k>1$.

A C^∞-function ϕ on R belongs to the class $C\{M_n\}$ if there is a positive number A_ϕ such that $\|\phi\|_\infty \leqq A_\phi$ and $\|\phi^{(k)}\|_\infty \leqq A_\phi^k M_k$, $k\geqq 1$.

The theory of C^∞-functions permits us to suppose, without loss of generality, that $\mu_k\geqq\mu_{k+1}$, $k\geqq 1$. Then the Denjoy-Carleman theorem [**3**, p. 376] tells us that the class $C\{M_n\}$ is a quasi-analytic class if, and only if, $\mu=\sum \mu_k<\infty$.

THEOREM. *There is a function f in $C\{M_n\}$ such that $f(x)=0 \Leftrightarrow x \in E$ and furthermore, for every $s \in S$, the order of the zero of f at s is $d(s)$.*

PROOF. We begin with some preliminaries. First, we apply Bray's construction to the characteristic function of the interval $[-\mu, \mu]$ to obtain a function M in $C\{M_n\}$ with $A_M=1$ and $M(x)>0 \Leftrightarrow x \in (-2\mu, 2\mu)$. Next, we notice that it suffices to suppose that each component of the complement of E has length <1. Then we observe that translating and stretching M yields a function g in $C\{M_n\}$ with $A_g=1/2$, $g(x)=0$ if $x\leqq 0$, and g strictly increasing (even convex if you wish) on (say) $[0, 3]$. For $a<b<a+1$, set $h(a, b, t)=g(t-a)g(b-t)$, $t \in R$. Then [**3**, pp. 373–374] $h(a, b, t)>0 \Leftrightarrow t \in (a, b)$ and $A_{h(a,b,t)}=1$. Now we are ready to begin construction of the requisite function f. The first step is to let $\{(a_i, b_i)\}$ be an enumeration of the components of $R-E$ and set $h(t)=\sum_i p_i h(a_i, b_i, t)$,

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where the p_i's have absolute value one. Then $h(t)=0 \Leftrightarrow t \in E$. Moreover, integrating that $(k+2)$nd derivative on (a_i, b_i) permits us to assert that there are constants P_k such that $|h^{(k)}(t)| \leqq P_k |(t-a_i)^2(b_i-t)^2$, $t \in (a_i, b_i)$, $k=1, 2, \cdots$. Hence, $h^{(1)} \equiv 0$ on E; and if we suppose that $h^{(k)}(t)=0$ for all $t \in E$, then it follows that $h^{(k+1)} \equiv 0$ on E. Thus $h \in C\{M_n\}$ and $A_h=1$. If $S=\varnothing$, we are done; otherwise, the second step is to adjust the order of zero at the isolated points of E. To this end we recall that if $s \in S$, the two components, (a_i, s) and (s, b_j), of $R-E$ abut at s. Then we denote by (α_s, β_s) the segment of length $\frac{1}{2}\min\{s-a_i, b_j-s\}$ centered at s. Next we notice that since $n\mu_n \to 0$, there is a constant $Q_s > 0$ such that the function, g_s, defined by $g_s(t)=Q_s(t-s)^{d(s)}h(\alpha_s, \beta_s, t)$ has order $d(s)$ at s, is zero on a neighborhood of $E-\{s\}$ and belongs to $C\{M_n\}$ with $A_{g_s}=1$. Set $y(t)= \sum_s q_s g_s(t)$, where $|q_s|=1$. One final adjustment is necessary; it remains to specify the p_i's and q_s's so that $f=h+y$ is nonzero on $R-E$ as follows. Enumerate S. Set $q_{s_1}=1$. Specify q at abuting points of S so that y is either nonnegative or nonpositive on each (a_i, b_i). Iterate this procedure as far as possible. Either q is defined on all of S or there is a first s_j not specified. Then set $q_{s_j}=1$ and repeat the process. In this way, all the q_s's are specified. Set $p_i=-1$ if y is negative at some point of (a_i, b_i) and $p_i=1$ otherwise.

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