

SUBALGEBRAS OF $B[c]$

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ABSTRACT. Two classes of subalgebras of the bounded operators on the Banach space of convergent sequences are studied. One class contains the well-known algebra of conservative matrices and the other contains the algebra of almost matrices. It is shown that the nontrivial members within each class are isomorphic to each other.

We denote the Banach spaces of convergent and bounded complex sequences by c and m , respectively, and the Banach algebra of bounded linear operators on c (with the usual uniform norm) by $B[c]$. With \lim denoting the functional $\lim x = \lim_i x_i$ on c and with e and e^k denoting, respectively, the sequences $(1, 1, 1, \dots)$ and $(0, \dots, 0, 1, 0, \dots)$, where 1 appears in the k th coordinate and zeros elsewhere, $k = 1, 2, \dots$, we have the important functionals χ and χ_i on $B[c]$ defined by (see [6, p. 241]) $\chi(T) = \lim(Te) - \sum_k \lim(Te^k)$ and $\chi_i(T) = (Te)_i - \sum_k (Te^k)_i$, $i = 1, 2, \dots$. (All sequence subscripts, as well as all indices of summation, run from 1 to ∞ .)

As usual we identify c^{**} (the second dual space of c) with m . Then T^{**} , the second adjoint of an operator T in $B[c]$, is a mapping from m to m and has the following matrix representation:

$$T^{**} = \begin{bmatrix} \chi(T) & b_1 & b_2 & b_3 & \cdots \\ \chi_1(T) & b_{11} & b_{12} & b_{13} & \cdots \\ \chi_2(T) & b_{21} & b_{22} & b_{23} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

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where $b_{nk} = (Te^k)_n$ and $b_k = \lim(Te^k)$. For further discussion of this representation and some of the remarks that follow, see, for example, [3] (and [1]). For each T in $B[c]$ we have the unique representation $T = v \otimes \lim + B$, where B is the matrix (b_{nk}) , v is the bounded sequence $(\chi_i(T))$, and $v \otimes \lim$ denotes the one-dimensional operator from c to m which sends x to $(\lim x)v$. Let

$$\Omega = \left\{ T \in B[c] : \lim_i \chi_i(T) \text{ exists} \right\}$$

and let

$$\Gamma = \{ T \in B[c] : \chi_i(T) = 0 \text{ for each } i \}.$$

Then Ω and Γ are proper subalgebras of $B[c]$ with $\Omega \supset \Gamma$. Γ is the well-known algebra of conservative matrices and χ is a scalar homomorphism on Γ . Those members T in Γ for which $\chi(T) = 0$ are called conull and the algebra of conull matrices is denoted by Ψ . These and other subalgebras of $B[c]$ were studied in [1]. It was shown there that χ is the only nonzero scalar homomorphism on Γ , that Ω also supports only one nonzero scalar homomorphism denoted by ρ and defined by $\rho(T) = \chi(B)$, and that Ψ , ρ^\perp (the kernel of ρ), Γ , and Ω are the only proper subalgebras of $B[c]$ that contain Ψ ($= \rho^\perp \cap \Gamma$).

More recently, A. Wilansky [5] introduced the following two classes of subalgebras of $B[c]$. (Our notation differs slightly from his so as to agree with the notation in [1].) Let \hat{c} denote the canonical embedding of c in c^{**} . (\hat{c} is identified with the set of sequences converging to their first term. See, for example, [7, p. 102].) Given any w in c^{**} ($=m$) let

$$\Gamma_w = \{ T \in B[c] : T^{**}w = \lambda w \text{ for some scalar } \lambda \}$$

and let

$$\Omega_w = \{ T \in B[c] : T^{**}w \in w \oplus \hat{c} \},$$

where by $w \oplus X$ (w a vector and X a linear space) we mean the linear span $\{\lambda w + x : x \in X \text{ and } \lambda \text{ a scalar}\}$. Clearly, Γ_w is contained in Ω_w . If $w \notin \hat{c}$, then for $T \in \Omega_w$ (resp., $T \in \Gamma_w$), $T^{**}w = \lambda w + \hat{x}$ (resp., $T^{**}w = \lambda w$) thus defining a functional ρ_w on Ω_w (resp., on Γ_w) by $\rho_w(T) = \lambda$. It is proved in [5, p. 356] that Ω_w is a closed subalgebra of $B[c]$ containing the compact operators, that Γ_w is a closed subalgebra of Ω_w , and that ρ_w is a scalar homomorphism on Ω_w and on Γ_w for $w \notin \hat{c}$. It is also shown that $\Omega_w = \Omega$, $\Gamma_w = \Gamma$, and $\rho_w = \rho$ when $w = e^1$.

In this paper we study these subalgebras in more detail. For example, we show that when $w \notin \hat{c}$, Ω_w (resp., Γ_w) is either Ω (resp., Γ) or is algebraically isomorphic with Ω (resp., Γ). It follows from this that all of the relationships between $B[c]$, Ω , ρ^\perp , and Γ proved in [1] equally apply to $B[c]$, Ω_w , ρ_w^\perp , and Γ_w . We also show that $\bigcap \{ \Gamma_w : w \in m \} = \{ \lambda I : \lambda \text{ a scalar} \}$

and that $\bigcap \{\Omega_w : w \in m\} = I \oplus K$, where I denotes the identity operator in $B[c]$ and K denotes the set of all compact operators in $B[c]$.

We begin by listing some useful facts.

1. LEMMA. (a) \hat{c} may be identified with the set of sequences converging to their first term.

(b) $c = x \oplus \hat{c}$ for any convergent sequence x which does not converge to its first term.

(c) If $z \notin w \oplus \hat{c}$ then $(w \oplus \hat{c}) \cap (z \oplus \hat{c}) = \hat{c}$.

PROOF. These statements are well known. Part (a), for example, is done in [7, p. 102].

2. LEMMA. (a) If $w \notin \hat{c}$ and $z \in (w \oplus \hat{c}) \setminus \hat{c}$ then $\Omega_z = \Omega_w$.

(b) If $z = \lambda w$ with $\lambda \neq 0$ then $\Gamma_z = \Gamma_w$.

PROOF. In part (a) we have $z = \lambda w + \hat{x}$ with $\lambda \neq 0$ and $\hat{x} \in \hat{c}$. If $T \in \Omega_z$ then $T^{**}z = \rho_z(T)_z + \hat{y}$ with $\hat{y} \in \hat{c}$, and so $T^{**}w = (1/\lambda)T^{**}(z - \hat{x}) = (1/\lambda)(\rho_z(T)z + \hat{y} - T^{**}\hat{x}) \in w \oplus \hat{c}$. Hence, $\Omega_z \subset \Omega_w$. The reverse containment is proved similarly, as is part (b).

3. LEMMA. (a) $w \in \hat{c}$ if and only if $\Omega_w = B[c]$.

(b) If $w \in \hat{c}$ then $\Gamma_w \neq \Gamma$.

PROOF. The first half of (a) follows from the fact that \hat{c} is invariant under every T^{**} . To prove the second half assume $w \notin \hat{c}$. Then either w converges with $\lim w \neq w_1$ (Lemma 1(a)), or w is divergent. In the first case let $v = (-1, 0, -1, 0, \dots)$ and define $B = (b_{nk})$ by setting $b_{2n-1, 2n-1} = 1$ ($n = 1, 2, \dots$) and $b_{nk} = 0$ otherwise. Then $T = v \otimes \lim + B$ belongs to $B[c]$ and $T^{**}w = (0, w_2 - w_1, 0, w_4 - w_1, 0, \dots) \notin c$. Since $c = w \oplus \hat{c}$ (Lemma 1(b)), T does not belong to Ω_w . In the second case choose a subsequence $(w_{k(n)})$ of w (with $k(1) > 1$) which converges to some number different from w_1 , and let $T = (t_{nk})$ be the subsequence selecting matrix defined by setting $t_{n, k-1} = 1$ when $k = k(n)$ ($n = 1, 2, \dots$) and 0 otherwise. Then $T \in B[c]$ and $T^{**}w = (w_1, w_{k(1)}, w_{k(2)}, \dots) \in c \setminus \hat{c}$ and so $T^{**}w$ does not belong to $w \oplus \hat{c}$. Hence we again have that $T \notin \Omega_w$.

To prove (b) simply observe that if w is the zero sequence then $\Gamma_w = B[c] \neq \Gamma$, while if $w \neq 0$ then $w' \otimes \lim$ belongs to Γ_w but does not belong to Γ , where $w' = (w_2, w_3, w_4, \dots)$.

The next lemma is Lemma 1.1 of [1]. We repeat it here for easy reference.

4. LEMMA. Let v and x be bounded sequences with v divergent. Then there exists a conull multiplicative matrix B such that $Bv = x$.

(A matrix B in Γ is conull and multiplicative if $\chi(B) = \chi_i(B) = \lim(Be^i) = 0$ for each $i = 1, 2, \dots$)

5. THEOREM. *Let $w \notin \hat{c}$. Then $\Omega_z = \Omega_w$ if and only if $z \in (w \oplus \hat{c}) \setminus \hat{c}$.*

PROOF. The second half of the theorem is Lemma 2(a). To prove the first half assume that $z \notin (w \oplus \hat{c}) \setminus \hat{c}$. If z belongs to \hat{c} use Lemma 3(a) to conclude that $\Omega_z = B[c] \neq \Omega_w$. If $z \notin \hat{c}$ then z must have a cluster point p different from z_1 . By considering $z - pe$, if necessary, we assume that $z_1 \neq 0$ and that z has a subsequence $(z_{k(n)})$ (with $k(1) > 1$) which converges to 0. (Notice that if $y = z - pe$ then $\Omega_y = \Omega_z$ by Lemma 2(a).) Now consider the corresponding subsequence $(w_{k(n)})$ of w . It either converges or it diverges. In the latter case we may choose a subsequence $(m(n))$ of $(k(n))$ so that $(w_{m(n)})$ converges. The subsequence $(z_{m(n)})$ still converges to zero. Thus, without loss of generality we may assume that $(w_{k(n)})$ converges. Moreover, by adding a constant sequence to w (and using Lemma 2(a)) we may also assume that $(w_{k(n)})$ converges to zero.

Let $T = (t_{nk})$ be the subsequence selecting matrix $t_{n,k-1} = 1$ if $k = k(n)$ ($n = 1, 2, \dots$); let v be the bounded sequence $(1, w_{k(1)}, 1, w_{k(2)}, \dots)$; define $B = (b_{nk})$ by $b_{2n-1, 2n-1} = -1$ ($n = 1, 2, \dots$); and set $S = v \otimes \lim + B$. Then

$$S^{**}(T^{**}w) = w_1(0, 1, w_{k(1)}, 1, w_{k(2)}, \dots) - (0, w_{k(1)}, 0, w_{k(3)}, \dots)$$

and

$$S^{**}(T^{**}z) = z_1(0, 1, w_{k(1)}, 1, w_{k(2)}, \dots) - (0, z_{k(1)}, 0, z_{k(3)}, \dots).$$

Now use Lemma 4 to get a conull multiplicative matrix R such that $Rv = (w_2 - w_1, w_3 - w_1, w_4 - w_1, \dots)$. Then $(RST)^{**}w = w_1w - w_1^2e - R^{**}\hat{y}$ and $(RST)^{**}z = z_1w - z_1w_1e - R^{**}\hat{x}$, where $\hat{y} = (0, w_{k(1)}, 0, w_{k(3)}, \dots) \in \hat{c}$ and $\hat{x} = (0, z_{k(1)}, 0, z_{k(3)}, \dots) \in \hat{c}$. Thus both $(RST)^{**}w$ and $(RST)^{**}z$ belong to $w \oplus \hat{c}$. Since $z_1 \neq 0$, $(RST)^{**}z$ does not belong to \hat{c} and so it cannot belong to $z \oplus \hat{c}$ (by Lemma 1(c)). Thus, $\Omega_w \neq \Omega_z$.

As an immediate consequence of Theorem 5, Lemma 1(b), and the fact that $\Omega = \Omega_{e^1}$ [5, Theorem 5] we get the following corollary.

6. COROLLARY. *$\Omega_z = \Omega$ if and only if $z \in c \setminus \hat{c}$.*

In order to set up the algebra isomorphism from $B[c]$ onto $B[c]$ mentioned in the introduction we need to know that for each $w \notin \hat{c}$ we can produce an isomorphism in $B[c]$ whose second adjoint sends w to e^1 . We do this in two steps. First, we produce an isomorphism whose second adjoint sends w to some x in $c \setminus \hat{c}$ (Lemma 7). Then we produce another isomorphism whose second adjoint sends x to e^1 (Lemma 8). Our construction in Lemma 7 is patterned after the one in [2, Lemma 1].

7. LEMMA. *Let $w \notin \hat{c}$. Then there exists an isomorphism T in $B[c]$ such that $T^{**}w \in c \setminus \hat{c}$.*

PROOF. Since w is not in \hat{c} we may choose a convergent subsequence $(w_{k(n)})$ of w (with $k(1) > 1$) so that $w_{k(n)} \neq w_1$ for each n and $\lim_n w_{k(n)} = l \neq w_1$. Let $B = (b_{nk})$ be defined as follows.

Set $b_{nn} = 1$ for every n .

If $n = k(i) + j < k(i+1) - 1$ for some $i = 1, 2, 3, \dots$, and some $j = 0, 1, 2, \dots$, set $b_{n, k(i)-1} = (w_{n+1} - l) / (w_1 - w_{k(i)})$.

Let $b_{nk} = 0$ for all other choices of n and k . Next define v as follows.

Set $v_n = 0$ for $1 \leq n < k(1)$ and for $n = k(i) - 1$, $i = 2, 3, 4, \dots$.

If $n = k(i) + j < k(i+1) - 1$ for some $i = 1, 2, 3, \dots$, and some $j = 0, 1, 2, \dots$, set $v_n = (l - w_{n+1}) / (w_1 - w_{k(i)})$.

Now let $T = v \otimes \lim + B$. By computing T^{**} it is easily verified that the matrix obtained from T^{**} by multiplying each of its nondiagonal elements by -1 is the two sided inverse of T^{**} and is the second adjoint of the operator $-v \otimes \lim + (2I - B)$. Thus T^{**} is an isomorphism from m onto m and hence T is an isomorphism in $B[c]$. It remains only to compute $T^{**}w$. Let t_i denote the number of terms in the sequence w between $w_{k(i)}$ and $w_{k(i+1)}$. Then $T^{**}w = (w_1, \dots, w_{k(1)}, l, \dots, l, w_{k(2)}, l, \dots, l, w_{k(3)}, l, \dots)$, where t_i l 's appear between $w_{k(i)}$ and $w_{k(i+1)}$. Since $\lim_i w_{k(i)} = l$ and $l \neq w_1$, $T^{**}w$ clearly belongs to $c \setminus \hat{c}$.

8. LEMMA. Let $x \in c \setminus \hat{c}$. Then there exists an isomorphism T in $B[c]$ such that $T^{**}x = e^1$.

PROOF. Suppose first that $x_1 \neq 0$. Since $x \in c \setminus \hat{c}$, $\lim x$ exists and does not equal x_1 . Let $l = x_1 - \lim x$. Let $v = (-1/(lx_1))(x_2, x_3, x_4, \dots)$ and let $B = (b_{nk})$ be the diagonal matrix $b_{nn} = 1/l$, $n = 1, 2, 3, \dots$. Then $T = v \otimes \lim + B$ is the isomorphism. Indeed, a simple computation shows that $T^{**}x = e^1$ and that $T^{-1} = u \otimes \lim + A$, where $u = (x_2, x_3, x_4, \dots)$ and $A = (a_{nk})$ is the diagonal matrix $a_{nn} = l$, $n = 1, 2, \dots$.

Next suppose that $x_1 = 0$. Then $\lim x = l \neq 0$. Let $k > 1$ be the first subscript such that $x_k \neq 0$. Let $T = v \otimes \lim + B$, where $v = (1/l)e^{k-1}$ and $B = (b_{nk})$ is the matrix defined by

$$\begin{aligned} b_{nn} &= -1/l && \text{for } n \geq k, \\ b_{n, k-1} &= x_{n+1}/(lx_k) && \text{for } n \geq k, \text{ and} \\ b_{nn} &= 1 && \text{for } n = 1, 2, \dots, k-2. \end{aligned}$$

(If $k=2$ ignore the last step.) Then $T^{**}x = e^1$ and T is an isomorphism with inverse $T^{-1} = u \otimes \lim + A$, where $u_{n-1} = x_n$ for each $n \geq k$ and where

$$\begin{aligned} a_{nn} &= -l && \text{for } n \geq k, \\ a_{n, k-1} &= x_{n+1} && \text{for } n \geq k-1, \text{ and} \\ a_{nn} &= 1 && \text{for } n = 1, 2, \dots, k-2. \end{aligned}$$

(Again, ignore the last step if $k=2$.)

9. THEOREM. *Let $w \notin \hat{c}$. Then there exists an isomorphism T in $B[c]$ such that $T^{**}w = e^1$.*

PROOF. Simply combine the two isomorphisms gotten in Lemmas 7 and 8.

Recall that $\rho_w^\perp = \{T \in \Omega_w : \rho_w(T) = 0\}$. ρ_w^\perp is clearly a subalgebra of Ω_w .

10. THEOREM. *Let $w \notin \hat{c}$. Then there exists an algebra isomorphism ϕ from $B[c]$ onto $B[c]$ such that,*

- (a) $\Omega_w = \phi(\Omega)$,
- (b) $\rho_w^\perp = \phi(\rho^\perp)$, and
- (c) $\Gamma_w = \phi(\Gamma)$.

PROOF. Let T be an isomorphism in $B[c]$ whose second adjoint sends w to e^1 . Define $\phi: B[c] \rightarrow B[c]$ by $\phi(S) = T^{-1}ST$. It is easily verified that ϕ is an algebra isomorphism from $B[c]$ onto $B[c]$ with inverse $\phi^{-1}(S) = TST^{-1}$. It remains only to verify the three assertions. We do this for (a); the proofs for (b) and (c) are similar. Let $S \in \Omega = \Omega_{e^1}$. Then $S^{**}e^1 = \lambda e^1 + \hat{x}$ with $\hat{x} \in \hat{c}$. Thus $(\phi(S))^{**}w = (T^{-1}ST)^{**}w = (T^{-1}S)^{**}e^1 = T^{-1**}(\lambda e^1 + \hat{x}) = \lambda w + T^{-1**}\hat{x} \in w \oplus \hat{c}$ and so $\phi(\Omega) \subset \Omega_w$. On the other hand, if $S \in \Omega_w$ then $S^{**}w = \mu w + \hat{y}$ for some $\hat{y} \in \hat{c}$, and so $(\phi^{-1}(S))^{**}e^1 = (TST^{-1})^{**}e^1 = (TS)^{**}w = T^{**}(\mu w + \hat{y}) = \mu e^1 + T^{**}\hat{y} \in e^1 \oplus \hat{c}$. Thus, $\Omega_w \subset \phi(\Omega)$.

Let K denote the set of compact operators in $B[c]$.

11. COROLLARY. $\bigcap \{\Omega_w : w \in m\} = I \oplus K$.

PROOF. If $w \in \hat{c}$ then $\Omega_w = B[c]$ (Lemma 3(a)) and so $\bigcap \{\Omega_w : w \in m\} = \bigcap \{\Omega_w : w \notin \hat{c}\}$. For each $w \notin \hat{c}$ we have $\Omega_w = \phi(\Omega)$, where ϕ is the algebra isomorphism constructed in Theorem 10. But $\Omega = I \oplus \rho^\perp$ and so $\Omega_w = \phi(\Omega) = I \oplus \phi(\rho^\perp) = I \oplus \rho_w^\perp$. Hence, $\bigcap \{\Omega_w : w \in m\} = I \oplus \bigcap \{\rho_w^\perp : w \in m\}$. Since each ρ_w^\perp contains K [5, Theorem 3] the proof will be complete when we show that $\bigcap \rho_w^\perp \subset K$. Thus let $T = v \otimes \lim + B \in \bigcap \rho_w^\perp$. (Notice that $v \in c$ and that $B \in \Gamma$ because $T \in \rho^\perp \subset \Omega$.) Then $T^{**}w \in \hat{c}$ for every bounded sequence w . This means that B is conull and sums every bounded sequence and hence must be compact by Schur's theorem. (See [4, p. 17].) Since $v \otimes \lim$ is also compact it follows that $T \in K$.

12. THEOREM. $\bigcap \{\Gamma_w : w \in m\} = \{\lambda I : \lambda \text{ a scalar}\}$.

PROOF. Let $T \in \bigcap \Gamma_w$. Then, in particular, $T \in \Gamma$ and so $\chi(T) = \rho(T)$, $\chi_i(T) = 0$ ($i = 1, 2, \dots$) and $T = (t_{nk})$ with column limits t_k ($k = 1, 2, \dots$). Let $w \in c \setminus \hat{c}$, say $w = \mu e^1 + \hat{x}$ with $\mu \neq 0$ and $\hat{x} \in \hat{c}$. By computing $T^{**}w$ and equating it to $\rho_w(T)w$ we get the following set of equations:

$$\sum_k t_k \hat{x}_{k+1} = 0 \quad \text{and} \quad \sum_k t_{nk} \hat{x}_{k+1} = \rho_w(T) \hat{x}_{n+1} \quad (n = 1, 2, \dots).$$

Since ρ is the only scalar homomorphism on $\Omega (= \Omega_w)$ [1], $\rho_w = \alpha\rho$ for some scalar α . But the proof of Lemma 2(a) shows that α must be 1. Hence, $\rho_w = \rho$ on Ω and the latter of the equations becomes

$$\sum_k t_{nk} \hat{x}_{k+1} = \rho(T) \hat{x}_{n+1} \quad (n = 1, 2, \dots).$$

In particular, taking $\hat{x} = e^k$ ($k = 2, 3, 4, \dots$) we get $t_{nk} = 0$ for $n \neq k$ and $t_{nn} = \rho(T)$. Thus $T = \rho(T)I$ and so $\bigcap \Gamma_w$ is contained in the set $\{\lambda I : \lambda \text{ a scalar}\}$. Since the reverse containment is obvious, the proof of the theorem is complete.

Our next result improves [5, Theorem 4].

13. THEOREM. $\Gamma_z = \Gamma$ if and only if $z = \mu e^1$ with $\mu \neq 0$.

PROOF. The second half of this theorem is contained in the statement of Lemma 2(b). To prove the first half, assume that $z \neq \mu e^1$ for any $\mu \neq 0$. Then $z_i \neq 0$ for some $i > 1$. Either this is the only nonzero entry in z , or else $z_j \neq 0$ for some j different from i . In the first case, let $T = (t_{nk})$ be the matrix with $t_{n,i-1} = 1$ for $n = 1, 2, 3, \dots$, and zeros elsewhere. Then $T^{**}z = z_i e^i \neq \lambda z$ for any λ ; hence, T belongs to Γ but not to Γ_z . In the second case, take $T = (t_{nk})$ to be the matrix with $t_{i-1,i-1} = 1$ and zeros elsewhere. Then $T^{**}z = z_i e^i \neq \lambda z$ for any λ and so we again have T in Γ but not in Γ_z .

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