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LOCAL AUTOMORPHISMS ARE DIFFERENTIAL OPERATORS ON SOME BANACH SPACES

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ABSTRACT. If E belongs to a certain category of Banach spaces (the B^∞-smooth spaces) which include Hilbert spaces and if F is any normed space, it is proved that any local linear automorphism of $C^\infty(E, F)$ is a differential operator. This generalizes a result of J. Peetre when $E=R^n$ and $F=R$.

1. A result of J. Peetre ([2], [3]) is the following characterization of linear partial differential operators:

A linear map T of $C^\infty(R^n, R)$ into $C^\infty(R^n, R)$ is a linear "partial differential operator" if and only if T is local i.e. for each $f \in C^\infty(R^n, R)$, $\operatorname{support}(Tf) \subset \operatorname{support}(f)$.

It should be noted that by a linear partial differential operator T is meant a collection $\{A_\alpha\} \subset C^\infty(R^n, R)$ such that the sets

$$G_\alpha = \{x \in R^n \mid A_\alpha(x) \neq 0\}$$

from a locally finite collection and such that $T(f)(x) = \sum_\alpha A_\alpha(x) D^\alpha(f)(x)$ for each $x \in R^n$ and each $f \in C^\infty(R^n, R)$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \sum_{i=1}^n \alpha_i$ and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}.$$

In this paper we prove that at least for E in a certain category of Banach spaces this theorem extends to local (linear) automorphisms on $C^\infty(E, F)$ where $C^\infty(E, F)$ now denotes the infinitely Fréchet differentiable F-valued functions on E and F is any normed linear space. Defining $L_s^k(E, F)$ to be the bounded symmetric k-multilinear maps from E to F we have $D^k f(x) \in L_s^k(E, F)$ for each x.

A natural generalization of a finite dimensional differential operator to

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an arbitrary Banach space is

$$T(f)(x) = \sum_{i=0}^{\infty} \alpha_i(x)(D^i f(x))$$

where $\alpha_i \in C^\infty(E, L(L_s^k(E, F), F))$ and the supports of the α_i form a locally finite collection. Such maps are clearly local linear automorphisms on $C^\infty(E, F)$.

As in Wells [6] we let

$$B^k(E, F) = \left\{ f \mid f \in C^k(E, F), \sup_{x \neq y} \| D^k f(x) - D^k f(y) \| / \| x - y \| < \infty \right\}$$

and

$$B^\infty(E, F) = \bigcap_{k=0}^{\infty} B^k(E, F).$$

An isomorphic invariant of a Banach space due to Bonic and Frampton [1] is C^p smoothness. E is C^p smooth, $p=0, 1, 2, \cdots, \infty$, if there is some $\eta \in C^p(E, R)$ with $\eta(0) \neq 0$ and $\eta(\{x \mid \|x\| \geqq 1\})=0$. Similarly, as in [6], E will be called B^p smooth, $j=0, 1, 2, \ldots, \infty$, if there is an $\eta \in B^p(E, R)$ with $\eta(0) \neq 0$ and $\eta(\{x \mid \|x\| \geqq 1\})=0$. B^∞ smooth spaces have been called uniformly C^∞ smooth in Quinn [4]. Finite dimensional spaces as well as $\mathscr{L}^p$ for p an even integer are B^∞ smooth. l^1 is not C^1 smooth. c_0 is C^∞ smooth but not B^1 smooth. Separable C^p or B^p smooth Banach spaces admit partitions of unity of class C^p or B^p respectively. In these cases $C^p(E, F)$ or $B^p(E, F)$ is dense in $C^0(E, F)$ or $B^0(E, F)$ respectively for any B-space F. Refer to Bonic and Frampton [1], Wells ([5] and [6]) for more details.

2. THEOREM. *If E and F are Banach space, if E is B^∞ smooth and if $T: C^\infty(E, F) \to C^\infty(E, F)$ is a local linear map, then T is a differential operator in the sense described above.*

The proof will require three lemmas. Only the first will use the B^∞ smoothness of E. We will use $K_r(x)$ to denote the open ball of radius r centered at x.

LEMMA 1. *Let $x_0 \in E$. There is a neighborhood U_{x_0} of x_0 and an integer k with that property that if $f, g \in C^\infty(E, F)$, $y \in U_{x_0}$ and $D^i f(y) = D^i g(y)$ for $0 \leqq i \leqq k$ then $T(f)(y) = T(g)(y)$.*

PROOF. If this were not the case there would be an $x_0 \in E$, a sequence x_n tending to x_0 and a sequence $f_n \in C^\infty(E, F)$ with $D^k f_n(x_n)=0$ for $k \leqq n$ and $\|T(f_n)(x_n)\| = n$. By the B^∞ smoothness there exists an $\eta \in B^\infty(E, R)$ with $\eta(\mathrm{cl}(K_{1/2}(0)))=1$ and $\eta(\{x \mid \|x\| \geqq 1\})=0$. Let $N_i = \sup_x \| D^i \eta(x) \|$. For each n there is an M_n and an r_n such that $\|D^j f_n(x)\| \leqq M_n(\|x-x_n\|)^{n+1-j}$ for $x \in K_{r_n}(x_n)$ and $0 \leqq j \leqq n$. Whenever $1/a_n < r_n$ we have

$$\begin{aligned}\sup_x \| D^j(f_n(x)\eta(a_n(x-x_n)))\| &\leqq \sum_{i=0}^{j} \binom{j}{i} \sup_{x \in K_{1/a_n}(x_n)} \| D^{j-i} f_n(x)\| \cdot a_n^i \cdot \sup_x \| D^i(\eta(x))\| \\ &\leqq \sum_{i=0}^{j} \binom{j}{i} M_n \cdot \left(\frac{1}{a_n}\right)^{n+1-j+i} \cdot a_n^i \cdot N_i = \left(\frac{1}{a_n}\right)^{n+1-j} M_n \sum_{i=0}^{j} \binom{j}{i} N_i.\end{aligned}$$

Thus we can choose a sequence a_n so that

(i) $1/a_n < r_n$,

(ii) $K_{1/a_n}(x_n) \cap K_{1/a_m}(x_m) = \varnothing$ for $n \neq m$,

(iii) $\sup_{j \leqq n, x \in E} \| D^j(f_n(x)\eta(a_n(x-x_n)))\| < \operatorname{dist}(x_0, K_{1/a_n}(x_n))$.

It follows that the function $f(x) = \sum_{n=1}^{\infty} f_n(x)\eta(a_n(x-x_n))$ belongs to $C^\infty(E, F)$ and that $f(x) \equiv f_n(x)$ for $x \in K_{1/2a_n}(x_n)$. Consequently

$$\|T(f)(x_n)\| = n$$

so that $T(f)$ is not a continuous function at x_0. This is a contradiction.

Let $E_k = F \oplus L_s^1(E, F) \oplus \cdots \oplus L_s^k(E, F)$. By Lemma 1 for each $x \in U_{x_0}$ there is a linear map $T_x : E_k \to F$ such that

$$T(f)(x) = T_x(f(x), Df(x), \cdots, D^k f(x)).$$

In Lemmas 2 and 3, x_0 and U_{x_0} will be fixed.

LEMMA 2. *T_x is bounded except possibly at a set I_{x_0} of isolated points of U_{x_0}.*

PROOF. If this were not the case there would exist a sequence $\{x_n\}$ with $\{x_n\} \subset U_{x_0}$ and a $y \in U_{x_0}$ with $y = \lim x_n$ and with T_{x_n} unbounded for each n. Next we choose a collection $\{\varphi_n\} \subset B^\infty(E, R)$ with support $\varphi_n \cap$ support $\varphi_m = \varnothing$ for $n \neq m$, $\operatorname{dist}(x_0, \text{support } \varphi_n) > 0$ for all n, and $\varphi_n(x) \equiv 1$ near x_n. (We observe that the B^∞ smoothness of E is not needed to construct the $\{\varphi_n\}$ since the x_n's can be separated by a disjoint collection of weak neighborhoods each of which is the support of a B^∞ function equal to 1 near x_n.) For each n choose $g_n \in C^\infty(E, F)$ such that

$$\sup_{j \leqq n, x \in K_1(x_0)} \| D^j(g_n(x)\varphi_n(x))\| < \operatorname{dist}(x_0, \text{support } \varphi_n)$$

and $\| T_{x_n}(\{g_n(x_n), \cdots, D^k g_n(x_n)\})\| \geqq n$. The function

$$f(x) = \sum_{n=0}^{\infty} g_n(x)\varphi_n(x)$$

belongs to $C^\infty(E, F)$ and $f(x)=g_n(x)$ near x_n. Consequently $\|T(f)(x_n)\| \geqq n$ which is impossible in view of the continuity of the function $T(f)$.

Thus T_x induces a map $T^0: U_{x_0}\backslash I_{x_0} \to L(E_k, F)$ such that $T^0(x)=T_x$. Hence $T^0(x)(f(x), \cdots, D^k f(x))=T(f)(x)$ for $x \in U_{x_0}\backslash I_{x_0}$.

LEMMA 3. *For each $p=0, 1, 2, \cdots$ and each $y_0 \in U_{x_0}$ there is a neighborhood U_{y_0} of y_0 such that $T^0|U_{y_0}\backslash I_{x_0} \in B^p(U_{y_0}\backslash I_{x_0}, L(E_k, F))$.*

PROOF. In Wells [6] it is shown that

$$B^p(E, F) = \left\{ f \,\middle|\, f \in C^0(E, F), \sup_{x,h\neq 0} \|\Delta_h^{p+1} f(x)\|/\|h\|^{p+1} < \infty \right\}$$

where $\Delta_h f(x)=f(x+h)-f(x)$. Suppose the lemma were false. Then for some p and some $y_0 \in U_{x_0}$ and for every neighborhood N of y_0 contained in x_0, the supremum of $\|\Delta_h^{p+1} f(x)\|/\|h\|^{p+1}$ over all x, $h\neq 0$ with x, $x+h, \cdots, x+(p+1)h$ contained in $N\backslash I_{x_0}$ would be infinite. This would imply the existence of sequences $\{x_n\}$, $\{h_n\}$ with $x_n \to y_0$, $h_n \to 0$,

$$\{x_n, x_n + h_n, \cdots, x_n + (p+1)h_n\} \in U_{x_0}\backslash I_{x_0}$$

and

$$\|\Delta_{h_n}^{p+1} T^0(x_n)\|/ \|h_n\|^{p+1} \geqq 4^n.$$

Choose $A_n \in E_k$ with $\|A_n\|_{E_k} \leqq 3^{-n}$ and

$$\|\Delta_{h_n}^{p+1} T^0(x_n)(A_n)\| \geqq \tfrac{3}{4} \|\Delta_{h_n}^{p+1} T^0(x_n)\| \cdot 3^{-n}.$$

Since for any t, s in a normed linear space $\sup\{\|t+\sigma s\| \,|\sigma=\pm 1\} \geqq \|s\|$, we may inductively choose $\sigma_n=\pm 1$, $n=1, 2, 3, \cdots$, so that

$$\left\|\Delta_{h_n}^{p+1} T^0(x_n)\left(\sum_{j=1}^{n} \sigma_j A_j\right)\right\| > \tfrac{3}{4} \|\Delta_{h_n}^{p+1} T^0(x_n)\| \cdot 3^{-n}.$$

For each n let g_n be the k polynomial such that $A_n=\{g_n(x_n), \cdots, D^k g_n(x_n)\}$ and $f(x)$ be the k polynomial $\sum_{i=1}^{\infty} \sigma_j g_j(x)$. Then

$$\begin{aligned}
\|\Delta_{h_n}^{p+1} T(f)(x_n)\| &= \|\Delta_{h_n}^{p+1} T^0(x_n)(f(x_n), \cdots, D^k f(x_n))\| \\
&\geqq \left\|\Delta_{h_n}^{p+1} T^0(x_n)\left(\sum_{j=1}^{n} \sigma_j A_j\right)\right\| - \left\|\Delta_{h_n}^{p+1} T^0(x_n)\left(\sum_{n+1}^{\infty} \sigma_j A_j\right)\right\| \\
&\geqq \|\Delta_{h_n}^{p+1} T^0(x_n)\| \cdot \left(\tfrac{3}{4}3^{-n} - \sum_{j=n+1}^{\infty} 3^{-j}\right) \\
&= \tfrac{1}{4}3^{-n} \cdot \|\Delta_{h_n}^{p+1} T^0(x_n)\| \geqq \tfrac{1}{4}(4/3)^n \cdot \|h_n\|^{p+1}.
\end{aligned}$$

But this is a contradiction since, for every p, $T(f)$ is B^p in some neighborhood of y_0. Q.E.D.

We are now in a position to prove the theorem. First observe that the choice of $p=0$ in Lemma 3 implies that the exceptional set I_{x_0} of Lemma 2 is void. Hence T^0 is defined on all of U_{x_0} and by Lemma 3 is locally B^p for any p so that $T^0 \in C^\infty(U_{x_0}, L(E_k, F))$. Consequently there exist $\alpha_n^0 \in C^\infty(U_{x_0}, L(L_s^n(E, F), F))$, $n=0, 1, \cdots, k$, such that

$$T(f)(x) = \sum_{n=0}^{k} \alpha_n^0(D^n f(x))$$

for all $x \in U_{x_0}$. Suppose that $T(f)(x)=\sum_{n=0}^{k'} \alpha_n'(D^n f(x))$ for $x \in U_{x_1}$ with $\alpha_n' \in C^\infty(U_{x_1}, L(L_s^n(E, F), F))$. Without loss of generality we may assume $k=k'$. If $x \in U_{x_0} \cap U_{x_1}$ and $A \in L_s^n(E, F)$ for $n \leqq k$, then for $g(x)=(1/n!) A(x, x, \cdots, x)$ we find $\alpha_n^0(A)=T(g)(x)=\alpha_n'(A)$. Hence on $U_{x_0} \cap U_{x_1}$, α_n^0 and α_n' agree, so that we may define maps

$$\alpha_n \in C^\infty(E, L(L_s^n(E, F), F)), \qquad n=0, 1, \cdots,$$

such that $(Tf)(x)=\sum_{n=0}^{\infty} \alpha_n(x)(D^n f(x))$ for $x \in E$ and the $\{\alpha_n\}$ have locally finite supports. Consequently T is a differential operator.

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