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A PRIME-DIVISOR FUNCTION

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ABSTRACT. This note studies the asymptotic mean values over arithmetical progressions, the general distribution of values, and the maximum order of magnitude, of a certain natural prime-divisor function of positive integers.

Consider the multiplicative arithmetical function β defined by $\beta(1)=1$ and $\beta(n)=\alpha_1\alpha_2\cdots\alpha_r$ if $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ (p_i prime, $\alpha_i>0$). Kendall and Rankin [**2**, p. 199] pointed out that this function has the finite *mean value*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\beta(n)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}=1.943\cdots$$

Strangely, perhaps, there appears to be virtually no other information available about this natural arithmetical function. (See note added in proof.) This note makes a more detailed study of its asymptotic properties.

1. Average values and distribution.

THEOREM 1. *Let r and q denote relatively prime positive integers. Then.*

$$\begin{aligned}\sum_{n\leq x;\, n\equiv r(\mathrm{mod}\ q)}\beta(n)=\frac{1}{q}\Bigg\{&\frac{L(2,\chi_0)L(3,\chi_0)}{L(6,\chi_0)}x+\frac{L(\frac{1}{2},\chi_0)L(\frac{3}{2},\chi_0)}{L(3,\chi_0)}x^{1/2}\\ &+\frac{L(\frac{1}{3},\chi_0)L(\frac{2}{3},\chi_0)}{L(2,\chi_0)}x^{1/3}+\frac{\bar{\chi}_1(r)L(\frac{1}{2},\chi_1)L(\frac{3}{2},\chi_1)}{L(3,\chi_0)}x^{1/2}\\ &+\sum_{\chi_2}\frac{\bar{\chi}_2(r)L(\frac{1}{3},\chi_2)L(\frac{2}{3},\chi_2^2)}{L(2,\chi_0)}x^{1/3}\Bigg\}\\ &+O(x^{3/10}\log^{9/10}x\,\phi(q)q^{8/5}),\end{aligned}$$

where χ_0 *denotes the principal character* mod q, *and the terms in* χ_1, χ_2 *occur if and only if there exist characters* $\chi_1\neq\chi_0$, $\chi_2\neq\chi_0$ mod q *such that* $\chi_1^2=\chi_0$, $\chi_2^3=\chi_0$.

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PROOF. Given any character χ mod q, the Euler product formula for $L(z,\chi)$ leads to the equation

$$\sum_{n=1}^{\infty} \chi(n)\beta(n)n^{-z} = \frac{L(z,\chi)L(2z,\chi^2)L(3z,\chi^3)}{L(6z,\chi^6)} \qquad [\mathrm{Re}\, z > 1].$$

Therefore, if $L(z,\chi)L(2z,\chi^2)L(3z,\chi^3) = \sum_{n=1}^{\infty} c(n,\chi)n^{-z}$ [Re $z>1$], then

$$\sum_{n\leq x} \chi(n)\beta(n) = \sum_{km^6\leq x} c(k,\chi)\chi^6(m)\mu(m) = \sum_{m\leq x^{1/6}} \chi^6(m)\mu(m) \sum_{k\leq x/m^6} c(k,\chi).$$

Now, by equations (68–71) of Richert [7], we have:

(i) $$\sum_{n\leq x} c(n,\chi_0) = \frac{\phi(q)}{q}\{xL(2,\chi_0)L(3,\chi_0) + x^{1/2}L(\tfrac{1}{2},\chi_0)L(\tfrac{3}{2},\chi_0) + x^{1/3}L(\tfrac{1}{3},\chi_0)L(\tfrac{2}{3},\chi_0)\} + O(x^{3/10}\log^{9/10} x\phi(q)q^{8/5});$$

(ii) $$\sum_{n\leq x} c(n,\chi_1) = \frac{\phi(q)}{q} x^{1/2}L(\tfrac{1}{2},\chi_1)L(\tfrac{3}{2},\chi_1) + O(x^{3/10}\log^{9/10} x\phi(q)q^{4/3})$$
if $\chi_1^2 = \chi_0$ but $\chi_1 \neq \chi_0$;

(iii) $$\sum_{n\leq x} c(n,\chi_2) = \frac{\phi(q)}{q} x^{1/3}L(\tfrac{1}{3},\chi_2)L(\tfrac{2}{3},\chi_2^2) + O(x^{3/10}\log^{9/10} x\phi(q)q^{3/5})$$
if $\chi_2^3 = \chi_0$ but $\chi_2 \neq \chi_0$;

(iv) $$\sum_{n\leq x} c(n,\chi) = O(x^{3/10}\log^{9/10} xq^{8/5}) \quad \text{for all other } \chi.$$

It follows for example that

$$\begin{aligned}\sum_{n\leq x} \chi_1(n)\beta(n) &= \sum_{m\leq x^{1/6}} \chi_1^6(m)\mu(m)\Big\{\frac{\phi(q)x^{1/2}}{qm^3} L(\tfrac{1}{2},\chi_1)L(\tfrac{3}{2},\chi_1) \\ &\qquad + O((x/m^6)^{3/10}\log^{9/10}(x/m^6)\phi(q)q^{4/3})\Big\} \\ &= \frac{\phi(q)}{q} x^{1/2}L(\tfrac{1}{2},\chi_1)L(\tfrac{3}{2},\chi_1)\left[\frac{1}{L(3,\chi_0)} + O(x^{-1/3})\right] \\ &\qquad + O(x^{3/10}\log^{9/10} x\phi(q)q^{4/3}) \\ &= \frac{\phi(q)}{q} x^{1/2} \frac{L(\tfrac{1}{2},\chi_1)L(\tfrac{3}{2},\chi_1)}{L(3,\chi_0)} + O(x^{3/10}\log^{9/10} x\phi(q)q^{4/3}).\end{aligned}$$

Similarly one can determine estimates for $\sum_{n\leq x}\chi(n)\beta(n)$ in the other cases. The theorem then follows from the equation

$$\sum_{n\leq x;\, n\equiv r (\mathrm{mod}\, q)} \beta(n) = \sum_{n\leq x} \beta(n) \frac{1}{\phi(q)} \sum_{\chi \mathrm{mod}\, q} \chi(n)\bar{\chi}(r).$$

COROLLARY.

$$\sum_{n\leqq x}\beta(n)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}x+\frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)}x^{1/2}+\frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)}x^{1/3}+O(x^{3/10}\log^{9/10}x).$$

REMARK. By using a theorem of Schmidt [5] concerning $\sum_{n\leqq x}c(n)$ where $\sum c(n)n^{-z}=\zeta(z)\zeta(2z)\zeta(3z)$ [Re $z>1$], one can sharpen the error term in this corollary to $O(x^{7/27}\log^2 x)$. For a more general, though less sharp, version of the corollary, see the author [3]. (See note added in proof.)

Next, by applying a theorem of Schoenberg [6, p. 319], one obtains

THEOREM 2. *The function β possesses an asymptotic distribution function*

$$F(x)=\lim_{N\to\infty}\frac{1}{N}\operatorname{card}\{n\leqq N:\beta(n)\leqq x\}.$$

This function is discrete, and the characteristic function of $F(e^x)$ is

$$\int_{-\infty}^{\infty}e^{itx}\,dF(e^x)=\prod_{\text{primes }p}(1-p^{-1})\left\{1+\sum_{r=1}^{\infty}p^{-r}e^{it\log r}\right\}.$$

REMARK. As part of a different discussion, J. Ridley and the author [4] have shown that, for each $k=1,2,\cdots$, the function β has a finite kth *moment*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}[\beta(n)]^k=\prod_{\text{primes }p}\left\{1+\sum_{r=2}^{\infty}[r^k-(r-1)^k]p^{-r}\right\}.$$

It may also be mentioned that a slight modification of a technique of Kendall and Rankin [2, p. 204] (who are concerned with the total number $a(n)$ of nonisomorphic abelian groups of order n) leads to an explicit formula for the *frequency*

$$F_m=\lim_{N\to\infty}\frac{1}{N}\operatorname{card}\{n\leqq N:\beta(n)=m\}.$$

Since this formula is relatively involved, we do not give it in detail, but we note that by combining certain of the frequencies P_m calculated for $a(n)$ in [2, p. 205] one obtains:

$$\begin{aligned}
&F_1=P_1=6/\pi^2=0.6079\cdots, && F_2=P_2=0.2008\cdots,\\
&F_3=P_3=0.0742\cdots, && F_4=P_4+P_5=0.0542\cdots,\\
&F_5=P_7=0.0147\cdots, && F_6=P_6+P_{11}=0.0215\cdots.
\end{aligned}$$

Thus $F_1+\cdots+F_6=0.9733\cdots$, which emphasizes how closely the values of β cluster about its mean value $1.943\cdots$. In fact, after a computer check on the "empirical" frequencies of β over the range $1 \leqq n \leqq 10{,}000$, J. Ridley has very kindly provided the following figures for the actual frequencies in this range: $F_1' = 0.6083$, $F_2' = 0.2008$, $F_3' = 0.0744$, $F_4' = 0.0541$, $F_5' = 0.0151$, $F_6' = 0.0216$. Here $F_1' + \cdots + F_6' = 0.9743$.

2. Maximum order of magnitude.

THEOREM 3. *Given any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that*

$$\beta(n) < 3^{(1/3)(1+\varepsilon)\log n/\log\log n} \quad \textit{for all } n \geqq n_0(\varepsilon),$$

while

$$\beta(n) > 3^{(1/3)(1-\varepsilon)\log n/\log\log n} \quad \textit{for infinitely many } n.$$

PROOF. The argument is parallel to one whereby Hardy and Wright [1] proved an analogous theorem for the divisor function $d(n)$. Firstly, one notes by induction that $a \leqq 3^{a/3}$ for $a = 1, 2, \cdots$. Hence, for $p \geqq 3^{1/3\delta}$ $(\delta > 0)$ and $a = 1, 2, \cdots$,

$$\frac{a}{p^{a\delta}} \leqq \frac{a}{3^{a/3}} \leqq 1.$$

If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ (p_i prime, $a_i > 0$), it follows that

$$\frac{\beta(n)}{n^\delta} = \prod_1^r \frac{a_i}{p_i^{a_i\delta}} \leqq \prod_{\text{primes } p \leqq 3^{1/3\delta}} (\delta \log 2)^{-1} < \exp\left(\frac{3^{1/3\delta}}{\delta \log 2}\right).$$

If $\delta = (1 + \frac{1}{2}\varepsilon)\log 3/3 \log\log n$ $(\varepsilon > 0)$, then

$$\frac{3^{1/3\delta}}{\delta \log 2} = \frac{(\log n)^{1/(1+\varepsilon/2)}\log\log n}{(1 + \frac{1}{2}\varepsilon)\log 2 \cdot \log 3^{1/3}} < \frac{\varepsilon \log 3^{1/3} \cdot \log n}{2 \log\log n}$$

for n sufficiently large. Hence the upper inequality follows.

For the lower inequality, let $N = (p_1 p_2 \cdots p_r)^3$ where $p_1 < \cdots < p_r$ denote the first r primes. Then

$$\log \beta(N) = r \log 3 = \pi(p_r)\log 3 \geqq \frac{\log(p_1 p_2 \cdots p_r) \cdot \log 3}{\log p_r},$$

in a similar way to the situation in [1, p. 263]. Hence, in the same way, there is a constant C such that

$$\log \beta(N) > \frac{\log N \cdot \log 3^{1/3}}{\log\log N + C} > \frac{(1 - \varepsilon)\log 3^{1/3} \cdot \log N}{\log\log N}$$

for N sufficiently large, i.e. for r sufficiently large.

THEOREM 4. *Given any $\varepsilon > 0$, $\beta(n) < 3^{(1/3)(1+\varepsilon)\log\log n}$ for "almost all" n, i.e. all n outside some set of asymptotic density zero.*

PROOF. The inequality $a \leqq 3^{a/3}$ $(a=1, 2, \cdots)$ implies that $\beta(n) \leqq 3^{\Omega(n)/3}$ where $\Omega(n)$ is the sum of the exponents of the prime divisors of n. The theorem then follows from Theorem 431 of [1], which states that $\Omega(n)$ has "normal order" $\log \log n$.

REMARK. One cannot expect a similar lower inequality, since for example β takes the value 1 on all square-free integers, and these have positive density $6/\pi^2$.

NOTE ADDED IN PROOF. In a recent paper, *The number of square-full divisors of an integer*, Proc. Amer. Math. Soc. **34** (1972), 79–80, D. Suryanarayana and R. Sita Rama Chandra Rao established the above corollary to Theorem 1, with an error estimate slightly weaker than that obtainable with the aid of Schmidt's theorem [5].

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