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CHEBYSHEV SUBSPACES AND CONVERGENCE OF POSITIVE LINEAR OPERATORS

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ABSTRACT. A theorem of Korovkin states that a sequence of positive linear operators on $C[a, b]$ converges strongly to the identity if and only if convergence holds on a three-dimensional Chebyshev subspace of $C[a, b]$. We extend this theorem to include Chebyshev subspaces of arbitrary dimension and convergence to other positive linear operators.

1. **Introduction.** Convergence of a sequence of positive linear operators to the identity can sometimes be proven by verifying convergence on a finite set of functions. For instance, let $C(X)$ denote the Banach space of real-valued functions which are defined and continuous on the interval $X=[a, b]$. Then a sequence of positive linear operators on $C(X)$ converges strongly to the identity provided convergence holds on a three-dimensional Chebyshev subspace of $C(X)$. This well known and striking result is due to Korovkin (cf. [**4**]).

There is another formulation of this theorem in terms of linear functionals on $C(X)$. Given an $x \in X$, we define point evaluation at x as the linear functional $\hat{x}(f)=f(x)$. A sequence of positive linear functionals $\{L_k\}$ converges weakly to $\hat{x}$, that is,

$$\lim_{k\to\infty} L_k(f) = \hat{x}(f), \qquad f \in C(X), \tag{1}$$

if and only if (1) holds for a three-dimensional Chebyshev subspace of $C(X)$.

The purpose of this paper is to extend this theorem to sequences of positive linear functionals converging to linear functionals other than point evaluations. Specifically, we answer the following question: What linear functionals L have the property that a sequence of positive linear functionals converges weakly to L on $C(X)$ if and only if convergence holds on an $n+1$-dimensional Chebyshev subspace of $C(X)$. We prove the corresponding result for positive linear operators, show that certain improvements in our result are not possible, and give some examples.

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For further generalizations of Korovkin's theorem see [**5**] or [**6**].

2. **Convergence of positive linear operators.** Consider the following positive linear functional

$$L(f) = \sum_{i=1}^{k} \lambda_i f(x_i) = \sum_{i=1}^{k} \lambda_i \hat{x}_i(f), \tag{2}$$

$\lambda_i > 0$, $i = 1, \cdots, k$ and $a \leqq x_1 < \cdots < x_k \leqq b$. Let us assign a weight of one to an x_i which lies in the interior of the interval $[a, b]$, while an endpoint, if it appears in (2), will be given a weight of $\frac{1}{2}$. The sum of these weights is called the index of L and will be denoted by $I(L)$. $\mathscr{L}_n$ is defined to be the space of all positive linear functionals of the form (2) which have index $\leqq n/2$. An n-dimensional linear subspace $\mathscr{U}$ of $C(X)$ is called a Chebyshev subspace if every nonzero element of $\mathscr{U}$ has at most $n-1$ zeros in X.

The basic ingredient in the proof of the Bohman-Korovkin theorem is the existence of nonnegative functions which have a prescribed set of zeros. The following result, whose proof can be found in [**2**], provides us with this information for an arbitrary Chebyshev subspace.

THEOREM A. *Let $\mathscr{U}$ be a $n+1$-dimensional Chebyshev subspace of $C(X)$ and suppose $S = \{s_1, s_2, \cdots, s_k\}$ is a set of k distinct points in $[a, b]$ such that $\sum_{i=1}^{k} w(s_i) \leqq n$ where $w(s)$ is defined to be two if $s \in (a, b)$ and one otherwise. Then there exists a nontrivial nonnegative function $u \in \mathscr{U}$ which vanishes precisely at the points of S. The only exception is that if $n = 2m$ and exactly one of the endpoints a or b is in S, then u may vanish at the other endpoint as well.*

THEOREM 1. *Let $\mathscr{U}$ be an $n+1$-dimensional Chebyshev subspace of $C(X)$. A sequence of positive linear functionals $\{L_k\}$ on $C(X)$ converges weakly to an $L \in \mathscr{L}_n$ if and only if*

$$\lim_{k \to \infty} L_k(u) = L(u), \qquad u \in \mathscr{U}. \tag{3}$$

PROOF. Assume that (3) holds. Since $\mathscr{U}$ is a Chebyshev subspace, there exists a $v \in \mathscr{U}$ which is positive on the interval $[a, b]$ (Theorem A). Set $m = \min_{x \in X} v(x)$, then in view of the positivity of L_k

$$\|L_k\| = L_k(1) \leqq L_k(v)/m. \tag{4}$$

Here, $\|L_k\|$ denotes the norm of the linear functional L_k. Since $\lim_{k\to\infty} L_k(v)$ exists, we conclude from (3) that $\operatorname{Sup}_k \|L_k\|$ is finite. The Helly selection theorem (cf. [**1**, Theorem 4.12.3]) states that any sequence of norm bounded linear functionals on $C(X)$ has a weakly convergent subsequence. Thus we conclude that $\{L_k\}$ has a weakly convergent subsequence.

Let L_0 denote any weak cluster point of the sequence $\{L_k\}$. The proof will be complete if we can show $L_0=L$. To this aim, we first note that L_0 is necessarily a positive linear functional which, on account of (3), satisfies

$$L_0(u) = L(u), \qquad u \in \mathscr{U}. \tag{5}$$

Furthermore, since $L \in \mathscr{L}_n$, L can be represented in the form

$$L = \sum_{i=1}^{k} \lambda_i \hat{x}_i, \qquad \lambda_i > 0, \tag{6}$$

with $I(L)\leqq n/2$. There exists, according to Theorem A, a nonnegative $\bar{u} \in \mathscr{U}$ which vanishes only at $x \in \{x_1, \cdots, x_k\}$ or possibly at an endpoint of $[a, b]$ not in $\{x_1, \cdots, x_k\}$. Therefore, from (5), (6) and the positivity of L_0, we conclude that

$$L_0 = \sum_{i=0}^{k} \mu_i \hat{x}_i, \tag{7}$$

where $\mu_i \geqq 0$, $i=0, 1, \cdots, k$ and $x_0 \in \{a, b\}$ (if $n \neq 2m$ or both endpoints are already in the set $\{x_1, \cdots, x_k\}$, then μ_0 can be taken to be zero).

Substituting this expression for L_0 into (5), we obtain the relation

$$\sum_{i=0}^{k} (\lambda_i - \mu_i)\hat{x}_i(u) = 0, \qquad u \in \mathscr{U} \tag{8}$$

where we have defined $\lambda_0=0$. Since $I(L)\leqq n/2$, there are no more than $n+1$ summands in (8) and so it follows that $\lambda_i=\mu_i$, $i=0, 1, \cdots, k$.

REMARK. Theorem 1 implies that the restriction of any $L \in \mathscr{L}_n$ to an $n+1$-dimensional Chebyshev subspace has a unique extension as a positive linear functional on $C(X)$.

We now turn our attention to the version of Theorem 1 which is valid for positive linear operators. Let T be a linear operator on $C(X)$. For each $x \in X$, we define the linear functional $\hat{x} \circ T$ by setting $(\hat{x} \circ T)f=(Tf)(x)$.

THEOREM 2. *Let T be a positive linear operator on $C(X)$ such that $\hat{x} \circ T \in \mathscr{L}_n$ for each $x \in X=[a, b]$. Then a sequence of positive linear operators T_k converges strongly to T if and only if*

$$\lim_{k\to\infty} T_k u = Tu, \qquad u \in \mathscr{U}, \tag{9}$$

where $\mathscr{U}$ is some $n+1$-dimensional Chebyshev subpace of $C(X)$.

PROOF. Suppose to the contrary that there exists a $g \in C(X)$ such that $T_k g$ does not converge to Tg while (9) holds. Then there exists an $\varepsilon_0>0$, $n_k\to\infty$ and a sequence $\{x_k\}\subseteq X$ such that

$$|(\hat{x}_k \circ T_{n_k})g - (\hat{x}_k \circ T)g| \geqq \varepsilon_0. \tag{10}$$

Since X is compact, the sequence $\{x_k\}$ has a convergent subsequence. We also denote this subsequence by $\{x_k\}$ and its limit by x_0. Then from (9) we have $\lim_{k\to\infty}(\hat{x}_k \circ T_{n_k})u=(\hat{x}_0 \circ T)u$, for all $u \in \mathcal{U}$. Hence, it follows from Theorem 1 that $\lim_{k\to\infty}(\hat{x}_k \circ T_{n_k})g=(\hat{x}_0 \circ T)g$. This contradicts (10) and so the theorem is proven.

EXAMPLE. Let $f \in C[0, 1]$. The nth Bernstein polynomial of f is defined by

$$(B_nf)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1 - x)^{n-k} f\left(\frac{k}{n}\right).$$

Note that B_n is a positive linear operator on $C[0, 1]$. Moreover, if we define $u_0(x)=1$, $u_1(x)=x$, and $u_2(x)=x(1-x)$ then the following equations can be verified

$$B_nu_0 = u_0, \qquad B_nu_1 = u_1, \quad \text{and} \quad B_nu_2 = (1 - 1/n)u_2 \tag{11}$$

(cf. [5]).

Let B_n^k denote the kth power of the linear operation B_n and define

$$Tf = f, \qquad \text{if} \lim_{n\to\infty} \frac{k_n}{n} = 0,$$

$$Tf = B_1f, \quad \text{if} \lim_{n\to\infty} \frac{k_n}{n} = \infty.$$

Then from (11) it immediately follows that $\lim_{n\to\infty} B_n^{k_n}Q \to TQ$ for any quadratic polynomial Q. Since $\hat{x} \circ T \in \mathcal{L}_2$ for every $x \in [0, 1]$ we can conclude from Theorem 2 that $\lim_{n\to\infty} B_n^{k_n}f \to Tf$ for all $f \in C[0, 1]$. This fact appears in [3].

3. **Improving Theorem 1.** We will show that Theorem 1 is not valid if either the hypothesis on L or the subspace $\mathcal{U}$ is removed.

THEOREM 3. *Suppose $\mathcal{U}$ is an $n+1$-dimensional Chebyshev subspace of $C(X)$. Let L be a positive linear functional on $C(X)$ with the property that its restriction to $\mathcal{U}$ has a unique extension as a positive linear functional on $C(X)$. Then L must be in $\mathcal{L}_n$.*

PROOF. Assume to the contrary that $L \notin \mathcal{L}_n$ and let $\{u_0, \cdots, u_n\}$ be a basis for the Chebyshev subspace $\mathcal{U}$. Consider the moment space $\mathcal{M}_{n+1}$ generated by the set of functions $\{u_0, u_1, \cdots, u_n\}$,

$$\mathcal{M}_{n+1} = \{c = (c_0, c_1, \cdots, c_n) \in E^{n+1} \mid c_i = F(u_i), i = 0, 1, \cdots, n\},$$

where F ranges over all positive linear functionals on $C(X)$. In [2] it is proven that $\mathcal{M}_{n+1}$ is a closed convex cone in E^{n+1} whose boundary is precisely the set

$$\{c = (c_0, \cdots, c_n) \in E^{n+1} \mid c_i = F(u_i),\ i = 0, 1, \cdots, n,\ F \in \mathscr{L}_n\}.$$

Thus, the vector $c=(L(u_0), \cdots, L(u_n))$ lies in the interior of $\mathscr{M}_{n+1}$. But for every interior point of $\mathscr{M}_{n+1}$ there exist exactly two positive linear functionals F_1 and F_2 in $\mathscr{L}_{n+1}$ which represent L, that is, for which we have $F_1(u)=F_2(u)=L(u)$, $u \in \mathscr{U}$. (This result also appears in **[2]**. F_1 and F_2 are referred to as principal representations of L.) This is a contradiction and so L must be in $\mathscr{L}_n$.

THEOREM 4. *Suppose $\mathscr{U}$ is an $n+1$-dimensional subspace of $C(X)$ which is not a Chebyshev subspace, then there exist positive linear functionals R and L on $C(X)$ such that $L \in \mathscr{L}_n$, $Ru=Lu$, for all $u \in \mathscr{U}$ and $R \neq L$.*

PROOF. Let $\{u_0, u_1, \cdots, u_n\}$ be a basis for the subspace $\mathscr{U}$. Since $\mathscr{U}$ is not a Chebyshev subspace, there exist points $t_0<t_1<\cdots<t_n$ such that the system of equations

$$\sum_{i=0}^{n} \lambda_i u_j(t_i) = 0, \qquad j = 0, 1, \cdots, n, \tag{12}$$

has a nonzero solution. We assume without loss of generality that $|\lambda_i|<1$, $i=0, 1, \cdots, n$ and the set $J=\{j:\lambda_j>0\}$ has cardinality $\leqq [n/2]$. Define $L=\sum_{j\in J} \lambda_j \hat{t}_j$ and $R=-\sum_{j\notin J} \lambda_j \hat{t}_j$. Then (12) implies $Lu=Ru$, for all $u \in \mathscr{U}$. Moreover, $L \in \mathscr{L}_n$, $L\neq R$ and R is a positive linear functional. This completes the proof.

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