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THE MODEL THEORY OF DIFFERENTIAL FIELDS OF CHARACTERISTIC $p \neq 0$

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ABSTRACT. The theory of differential fields of characteristic $p \neq 0$ is shown to have a model companion, the theory of differentially closed fields, which is moreover the model completion of the theory of differentially perfect fields. It is also shown that the theory of differentially closed fields is not ω-stable.

0. We consider the theory T_p of differential fields of characteristic $p \neq 0$, where p is a prime integer. This theory has a model companion T_p^*, the theory of differentially closed fields of characteristic p. We consider also an intermediate theory T'_p, such that T_p^* is the model completion of T'_p. The algebraic information needed to describe T_p^* is found in Seidenberg [7], and the procedure in defining T_p^* is analogous to that given for characteristic 0 by A. Robinson [5]. We also show that T_p^* is not ω-stable, in contrast to L. Blum's result that the theory of differentially closed fields of characteristic 0 is ω-stable.

Let the language L have similarity type with one binary relation $(=)$, two constants (0 and 1), three unary functions ($^{-1}$, $-$, and D), and two binary functions ($+$ and $\cdot$). The theory T_p is the usual theory of fields of characteristic p (in terms of $=$, 0, 1, $^{-1}$, $-$, $+$, and $\cdot$), together with the following two axioms for the derivative D:

$$\forall x \forall y (D(x \cdot y) = D(x) \cdot y + x \cdot D(y))$$

and

$$\forall x \forall y (D(x + y) = D(x) + D(y)).$$

The theory of fields of characteristic p in the given similarity type is universal, and the two axioms above are universal; thus T_p is a universal theory.

Let $\mathscr{F}$ be a model of T_p, with underlying field structure F. (In general we shall use script capitals for models of T_p and the corresponding Roman capitals for the underlying field.) An element $c \in F$ is a constant provided $D(c)=0$. The set of all constants in $\mathscr{F}$ is closed under $^{-1}$, $-$, $+$, $\cdot$, and D, and clearly 0 and 1 are constants. Therefore the set C of constants

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determines a submodel $\mathscr{C}$ of $\mathscr{F}$, called the constant sub(differential) field of $\mathscr{F}$. In particular, $F^p \subseteq C$, since $D(b^p)=0$ for any $b \in F$. The prime field F_p of characteristic p is contained in every model of T_p, and is moreover contained in the constant subfield of every model, since

$$D(1 + \cdots + 1) = D(1) + \cdots + D(1) = 0.$$

Thus T_p has a unique prime model $\mathscr{F}_p$, which is itself a constant field.

We remark that any field F of characteristic p gives rise to at least one model of T_p, where $D(a)=0$ for all $a \in F$. If F is perfect, then this is the only possible model of T_p; in particular all finite models of T_p are constant fields. This and other facts follow from the following standard result about extensions of differential fields. We state this result without proof; it is, for example, a special case of [**3**, Theorem 14, p. 172].

LEMMA 1. *Let $\mathscr{F} \models T_p$, and let $F'=F(b)$ be a field extension of F.*

(i) *If $b \notin F$, $b^p=a \in F$, and $D(a)=0$, then for any $c \in F'$ there exists a unique extension of D from F to F' such that $D(b)=c$ and such that the resulting structure $\mathscr{F}'$ is a model of T_p.*

(ii) *If b is separable algebraic over F, then there is exactly one way to extend D from F to F' such that $\mathscr{F}' \models \mathscr{F}_p$, $\mathscr{F}' \supseteq \mathscr{F}$.*

1. DEFINITION. A theory T has the amalgamation property provided whenever $\mathscr{F}$, $\mathscr{F}_1$, and $\mathscr{F}_2$ are models of T with $\mathscr{F} \subseteq \mathscr{F}_1$ and $\mathscr{F} \subseteq \mathscr{F}_2$, that there exists $\mathscr{F}_3 \models T$ such that $\mathscr{F}_1 \subseteq \mathscr{F}_3$ and $\mathscr{F}_2 \subseteq \mathscr{F}_3$.

THEOREM 2. *The theory T_p does not have the amalgamation property.*

PROOF. Let $F=F_p(t)$ be the field extension of the prime field by a transcendental element t, and let $\mathscr{F}$ be the corresponding constant differential field. Since t has no pth root in F, we can find an extension $F[c]$ of F such that $c \notin F$, $c^p = t$. By Lemma 1 there exist $\mathscr{F}_1 \supseteq \mathscr{F}$ and $\mathscr{F}_2 \supseteq \mathscr{F}$, both $\mathscr{F}_1$ and $\mathscr{F}_2$ models of T_p, with $F_1=F_2=F[c]$, such that $D(c)=0$ in $\mathscr{F}_1$ and $D(c)=1$ in $\mathscr{F}_2$.

If there exists $\mathscr{F}_3 \models T_p$ with $\mathscr{F}_1 \subseteq \mathscr{F}_3$ and $\mathscr{F}_2 \subseteq \mathscr{F}_3$, then $\mathscr{F}_3 \models D(c)= 0 \wedge D(c)=1$, which is impossible. Thus T_p does not have amalgamation.

We prove later that T_p possesses a model companion T_p^* (i.e., that there is T_p^* which is both model consistent relative to T_p and model complete). Using the following lemma we see that T_p has no model completion.

LEMMA 3 (ELI BERS [**2**]). *Let T be a theory which has a model companion T^*. Then T has a model completion if and only if T has the amalgamation property.*

COROLLARY. *The theory T_p has no model completion.*

PROOF. Immediate from Theorem 2 and Lemma 3.

Since the failure of amalgamation arises from the possible existence of constants without pth roots, the following is a reasonable extension of T_p.

Definition. The theory of differentially perfect differential fields of characteristic p, T'_p, is the theory T_p together with one additional axiom θ:

$$\theta = \forall x \exists y (D(x) = 0 \supset y^p = x).$$

Thus the models of T'_p are just the models of T_p which are closed with respect to extraction of pth roots of constants. Since T_p is universal and θ is $\forall\exists$, we have that T'_p is $\forall\exists$, but obviously not universal.

Theorem 4. *The theory T'_p is a model consistent extension of T_p.*

Proof. Let $\mathscr{F} \models T_p$, and let $\{a_\eta\}_{\eta<\alpha}$ be the set of all the constants of $\mathscr{F}$, indexed by some ordinal α. Define a chain $\{\mathscr{F}_\eta\}_{\eta<\alpha}$ of models of T_p as follows:

(i) $\mathscr{F}_0 = \mathscr{F}$.

(ii) If a_η has a pth root in $\mathscr{F}_\eta$, let $\mathscr{F}_{\eta+1} = \mathscr{F}_\eta$. If not, let $\mathscr{F}_{\eta+1} \supseteq \mathscr{F}_\eta$ such that $\mathscr{F}_{\eta+1}$ contains a pth root of a_η, with $\mathscr{F}_{\eta+1} \models T_p$. (Such an $\mathscr{F}_{\eta+1}$ exists, by Lemma 1.)

(iii) For $\beta<\alpha$, β a limit ordinal, let $\mathscr{F}_\beta = \bigcup_{\eta<\beta} \mathscr{F}_\eta$.

Now let $\mathscr{F}^{(1)} = \bigcup_{\eta<\alpha} \mathscr{F}_\eta$. Since T_p is universal (hence inductive), we have $\mathscr{F}^{(1)} \models T_p$; furthermore, every constant in $\mathscr{F}$ has a pth root in $\mathscr{F}^{(1)}$; repeating the above procedure ω times we obtain a chain $\mathscr{F} = \mathscr{F}^{(0)} \subseteq \mathscr{F}^{(1)} \subseteq \mathscr{F}^{(2)} \subseteq \cdots$, such that every constant in $\mathscr{F}^{(j)}$ has a pth root in $\mathscr{F}^{(j+1)}$ and such that $\mathscr{F}^{(j)} \models T_p$, $j=0, 1, \cdots$.

Finally, let $\mathscr{F}' = \bigcup_{j<\omega} \mathscr{F}^{(j)}$. Clearly $\mathscr{F}' \supseteq \mathscr{F}$, and $\mathscr{F}' \models T_p$, since T_p is inductive. If a is a constant in $\mathscr{F}'$, then for some $j<\omega$, a is in $\mathscr{F}^{(j)}$; hence a has a pth root in $\mathscr{F}^{(j+1)}$. Thus $\mathscr{F}' \models T'_p$, and we have shown that any model of T_p is contained in a model of T'_p as desired.

We observe that $\mathscr{F}_p \models T'_p$, since F_p is a perfect field, and so T'_p has the same prime model as does T_p.

2. In this section we describe the theory of differentially closed fields of characteristic p, which we call T^*_p. To find axioms for T^*_p we use Seidenberg's elimination theory [7], which enables us to axiomatize the class of existentially complete models of T_p.

We use the following notation:

(i) For integers $j, k>0$, we use $F_j(x_1, \cdots, x_k)$, $G_j(x_1, \cdots, x_k)$, and $H(x_1, \cdots, x_k)$ as abbreviations for terms in our language corresponding to polynomials in $x_1, \cdots, x_k$ with coefficients in F_p.

(ii) For $n>0$, we let $R(x_1, \cdots, x_n)$ be an abbreviation for a formula

$$\begin{aligned}\exists x_{n+1} \cdots \exists x_{n+t}(D(F_1(x_1, \cdots, x_n)) = 0 \wedge D(F_2(x_1, \cdots, x_{n+1})) = 0 \\ \wedge \cdots \wedge D(F_t(x_1, \cdots, x_{n+t-1})) = 0 \\ \wedge x_{n+1}^p = F_1(x_1, \cdots, x_n) \wedge x_{n+2}^p = F_2(x_1, \cdots, x_{n+1}) \\ \wedge \cdots \wedge x_{n+t}^p = F_t(x_1, \cdots, x_{n+t-1}) \wedge G_1(x_1, \cdots, x_{n+t}) = 0 \\ \wedge \cdots \wedge G_s(x_1, \cdots, x_{n+t}) = 0 \wedge H(x_1, \cdots, x_{n+t}) \neq 0),\end{aligned}$$

where $t, s>0$, and the F_j, G_j and H are as in (i).

Seidenberg proves that given a finite system of differential equations and inequations over F_p in variables $x_1, \cdots, x_{n+m}$:

$$\begin{gathered} f_1(x_1, \cdots, x_{n+m}) = 0 \\ \cdot \\ \cdot \\ \cdot \\ f_k(x_1, \cdots, x_{n+m}) = 0 \\ g(x_1, \cdots, x_{n+m}) \neq 0 \end{gathered}$$

that there exists a finite set of formulas

$$\{R_1(x_1, \cdots, x_n), \cdots, R_r(x_1, \cdots, x_n)\}$$

(where the R_j are as in (ii)), with the following property: for all $\mathscr{F} \models T_p'$ and all $a_1, \cdots, a_n \in \mathscr{F}$, statements (*) and (**) are equivalent:

(*) There exists $\mathscr{F}_1 \supseteq \mathscr{F}$, $\mathscr{F}_1 \models T_p$, and $b_1, \cdots, b_m \in \mathscr{F}_1$, such that

$$\begin{gathered} f_1(a_1, \cdots, a_n, b_1, \cdots, b_m) = 0 \\ \cdot \\ \cdot \\ \cdot \\ f_k(a_1, \cdots, a_n, b_1, \cdots, b_m) = 0 \\ g(a_1, \cdots, a_n, b_1, \cdots, b_m) \neq 0. \end{gathered}$$

(**) For some j, $1 \leqq j \leqq r$, $\mathscr{F} \models R_j(a_1, \cdots, a_n)$.

For each system $f_1, \cdots, f_k, g$ we let $\varphi_{f_1,\cdots,f_k,g,n,m}$ be the following sentence:

$$\begin{aligned}\varphi_{f_1,\cdots,f_k g,n,m} &= \forall x_1 \cdots \forall x_n((\exists x_{n+1} \cdots \exists x_{n+m} (f_1(x_1, \cdots, x_{n+m}) = 0 \\ &\wedge \cdots \wedge f_k(x_1, \cdots, x_{n+m}) = 0 \wedge g(x_1, \cdots, x_{n+m}) \neq 0) \\ &\Leftrightarrow (R_1(x_1, \cdots, x_n) \vee \cdots \vee R_r(x_1, \cdots, x_n))).\end{aligned}$$

The sentence φ says roughly that the $\mathscr{F}_1$ in (*) may be chosen to be $\mathscr{F}$ itself.

Finally we let $T_p^* = T'_p \cup \{\varphi_{f_1,\cdots,f_k,g,n,m} | f_1, \cdots, f_k, g$ is a system of differential equations over F_p in the variables $x_1, \cdots, x_{n+m}$, for some $n, m \geqq 0\}$.

In his elimination procedure, Seidenberg also proves that for a given $\mathscr{F} \models T'_p$, if $\mathscr{F}_1$ exists as in (∗), then $\mathscr{F}_1$ may be chosen to be an extension of any given model of T_p which contains $\mathscr{F}$. This translates in our terminology to the following.

LEMMA 5. *The theory T'_p has the amalgamation property.*

THEOREM 6. *The theory T_p^* is a model consistent extension of T'_p.*

PROOF. By Lemma 5 we may successively adjoin solutions to systems of differential equations and inequations to a given differentially perfect differential field $\mathscr{F}$. Therefore there exists a chain $\mathscr{F} = \mathscr{F}^{(0)} \subseteq \mathscr{F}^{(1)} \subseteq \cdots$ of models of T_p, such that any finite system of equations and inequations over $\mathscr{F}^{(j)}$, which has a solution in some extension of $\mathscr{F}^{(j)}$ has a solution in $\mathscr{F}^{(j+1)}$. Therefore $\mathscr{F}' = \bigcup_{j<\omega} \mathscr{F}^{(j)}$ is a model of T_p^*, since any finite system over $\mathscr{F}'$ is a finite system over $\mathscr{F}^{(j)}$ for some j, and if the system has a solution in any extension of $\mathscr{F}^{(j)}$, it has a solution in $\mathscr{F}^{(j+1)}$, hence in $\mathscr{F}'$, as desired.

THEOREM 7. *The theory T_p^* is model complete.*

PROOF. Any primitive formula in our language is equivalent to a finite system of equations and inequations over F_p; thus the class of models of T_p^* is exactly the class of existentially complete models of T_p. By Robinson's test, this is sufficient for T_p^* to be model complete.

THEOREM 8. *The theory T_p^* is the model completion of T'_p and is the model companion of T_p.*

PROOF. By Theorems 6 and 7, T_p^* is the model companion of T'_p, hence also of T_p by Theorem 4. Using Lemmas 3 and 5, we see T_p^* is the model completion of T'_p.

THEOREM 9. *The theory T_p^* is complete.*

PROOF. The model completion of a theory with a prime model must be a complete theory, by Theorem 4.2.3 of **[4]**. Since $\mathscr{F}_p$ is the prime model of T'_p, we conclude that T_p^* is complete.

We observe that the models of T_p^* are not algebraically closed fields, unlike the characteristic 0 differentially closed fields, since a nonconstant can have no pth root. We summarize a few observations about models of T_p^* in the following theorem.

THEOREM 10. *Let $\mathscr{F} \models T_p^*$, and let $\mathscr{C}$ be the constant subfield of $\mathscr{F}$. Then both $\mathscr{F}$ and $\mathscr{C}$ are separably algebraically closed (as fields), and $\mathscr{C}$ contains the algebraic closure of $\mathscr{F}_p$, the prime field.*

PROOF. If b is separable algebraic over $\mathscr{F}$, then by Lemma 1 there exists $\mathscr{F}' \supseteq \mathscr{F}$ such that b is in $\mathscr{F}'$. By the model completeness of T_p^*, this implies that b is in $\mathscr{F}$, hence $\mathscr{F}$ is separably algebraically closed.

If b is separable algebraic over $\mathscr{C}$, then b is also separable algebraic over $\mathscr{F}$, hence is in $\mathscr{F}$. Let $f(x)=x^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0$ be the minimum polynomial for b over $\mathscr{C}$. Taking the derivative of both sides of the equation $b^n+c_{n-1}b^{n-1}+\cdots+c_1b+c_0=0$ gives us

$$(nb^{n-1} + (n-1)c_{n-1}b^{n-2} + \cdots + c_1)D(b) = 0,$$

since $D(c_i)=0$ for $i=0, \cdots, n-1$.

Since b is separable over $\mathscr{C}$, $nb^{n-1}+(n-1)c_{n-1}b^{n-2}+\cdots+c_1 \neq 0$. Therefore $D(b)=0$ and b is in $\mathscr{C}$, as desired. Now $\mathscr{F}_p \subseteq \mathscr{C}$, and the separable algebraic closure of $\mathscr{F}_p$ is the algebraic closure of $\mathscr{F}_p$ (since $\mathscr{F}_p$ is perfect). Therefore $\mathscr{C}$ contains the algebraic closure of $\mathscr{F}_p$, and our proof is finished.

3. In the language L it is known that the theory of differentially closed fields of characteristic 0 admits elimination of quantifiers. While the theory T_p^* does not admit elimination of quantifiers for L, we can modify our language by adding one unary function symbol so that the resulting theory does have elimination of quantifiers. Let the language $\tilde{L}$ be obtained by adding a new unary function r to L, and let $\tilde{T}_p$ be the theory T_p together with the axiom

$$\varphi = \forall x \forall y ((r(x) = y \wedge D(x) = 0 \supset y^p = x) \wedge (D(x) \neq 0 \supset r(x) = 0)).$$

This restricts the notion of differential field to that of differentially perfect differential field, and the theory $\tilde{T}_p^* = T_p^* \cup \{\varphi\}$ is the model completion of $\tilde{T}_p$, by the same argument as before. Since the model completion of a universal theory always admits elimination of quantifiers, and $\tilde{T}_p$ is universal, we have the following

THEOREM 11. *The theory $\tilde{T}_p^*$ admits elimination of quantifiers.*

L. Blum gives a simple set of axioms for differentially closed fields of characteristic 0, and shows more generally that any model completion of a universal theory can be axiomatized by sentences involving only one existential quantifier (see **[1]** or **[6]**). Seidenberg shows in **[7]** that it is impossible to eliminate variables one by one in a system of differential equations and inequations over a differentially perfect differential field. To reduce a system to one involving one variable we must use the unary function r; a system in one variable can be further reduced to one equation and one inequation, as in the characteristic 0 case, but this pair must satisfy certain separability requirements in order to have a solution in some extension field. Thus, while axioms for $\tilde{T}_p^*$ involving only one existential quantifier must exist, Seidenberg's procedure does not provide a simple formulation of such axioms.

4. Our final observation also arises from seeking an analogy with Blum's work for characteristic 0. Blum proves that the theory of differentially closed fields of characteristic 0 is ω-stable. It follows that every differential field of characteristic 0 is contained in a prime differentially closed field (called its differential closure), which is furthermore unique by a theorem of Shelah (see [**6**] for details). However, this procedure gives us no information for characteristic p, as the following shows.

THEOREM 12. *The theory T_p^* is not ω-stable.*

PROOF. (We mean by the above that there exists a countable model of T_p^* over which there are uncountably many 1-types.)

Let $\mathscr{F} \models T_p^*$, $\mathscr{F}$ countable, and let $a \in \mathscr{F}$ with $D(a)=1$. Let c be (algebraically) transcendental over $\mathscr{F}$, and let $\mathscr{F}_1$ be the differential field with field structure $F_1=F(c)$, where $D(c)=1$ and $\mathscr{F}_1$ is an extension of $\mathscr{F}$.

Consider the element $c-a \in F(c)$. Since $D(c-a)=0$, it is possible for $c-a$ to have a pth root in some extension of $\mathscr{F}_1$.

If there were $b \in F_1$ with $b^p=c-a$, then b is of the form

$$b = \frac{\alpha_0 + \alpha_1 c + \cdots + \alpha_n c^n}{\beta_0 + \beta_1 c + \cdots + \beta_m c^m},$$

for some α_i, $\beta_j \in F$, where α_0 and β_0 are not both 0. Taking pth powers of the above equation gives us

$$\alpha_0^p + \alpha_1^p c^p + \cdots + \alpha_n^p c^{np} = (\beta_0^p + \cdots + \beta_m^p c^{mp})(c-a).$$

Regarding this as an equation in c over F and matching constant terms yields $\alpha_0^p = -\beta_0^p a$. But then $(\alpha_0/\beta_0)^p = -a$, and $D(-a)=0$, a contradiction. Thus $c-a$ has no pth root in F_1.

By Lemma 1 we may adjoin to F_1 a pth root c_1 of $c-a$ and define $D(c_1)$ arbitrarily; in particular we may take $D(c_1)=k_1$ for any integer k_1, $0 \leqq k_1 < p$. This determines an extension $\mathscr{F}_2$ of $\mathscr{F}_1$, with $F_2=F_1[c_1]$.

Now we claim c_1-k_1a is a constant without a pth root in F_2. For if $b \in F_2$ were such that $b^p=c_1-k_1a$, then by writing

$$b = \alpha_0 + \alpha_1 c_1 + \cdots + \alpha_{p-1} c_1^{p-1}, \qquad \alpha_i \in F_1,$$

we have

$$b^p = c_1 - k_1 a = \alpha_0^p + \alpha_1^p(c-a) + \cdots + \alpha_{p-1}^p(c-a)^{p-1} \in F_1,$$

and so $c_1 \in F_1$, a contradiction. Thus we can continue, adjoining a pth root c_2 of $c_1 - k_1 a$ and assigning $D(c_2) = k_2$ for some k_2, $0 \leqq k_2 < p$. This determines $\mathscr{F}_3$, an extension of $\mathscr{F}_2$, with $F_3 = F_2[c_2]$, and with $D(c_2 - k_2 a) = 0$ but with no pth root of $c_2 - k_2 a$ in $\mathscr{F}_3$.

In this manner, given $k_1, k_2, k_3, \cdots, 0 \leqq k_i < p$, there exist $\mathscr{F} \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_3 \subseteq \cdots$ and $c, c_1, c_2, \cdots$ such that

(i) $D(c) = 1$, $c \in F_1$, $c_1^p = c - a$,

(ii) $D(c_i - k_i a) = 0$, $D(c_i - ja) \neq 0$ for $0 \leqq j < p, j \neq k_i$, and

(iii) $D(c_i) = k_i$, $c_i \in F_{i+1}$, $c_{i+1} = c_i - k_i a$ for $i = 1, 2, \cdots$.

We have now the following set of formulas with free variable y:

$$D(y) = 1$$

$$\exists x_1 (x_1^p = y - a \wedge D(x_1) = k_1)$$

$$\exists x_1 \exists x_2 (x_1^p = y - a \wedge D(x_1) = k_1 \wedge x_2^p = x_1 - k_1 a \wedge D(x_2) = k_2)$$

$$\vdots$$

$$\exists x_1 \cdots \exists x_n (x_1^p = y - a \wedge D(x_1) = k_1 \wedge \cdots \wedge x_n^p = x_{n-1} - k_{n-1} a \wedge D(x_n) = k_n).$$

This set is extendable to a 1-type over $\mathscr{F}$; for each choice of $k_1, k_2, \cdots$ we obtain a distinct 1-type. Thus uncountably many 1-types exist over $\mathscr{F}$, and T_p^* is not ω-stable.

This leaves unanswered the question of whether there exists a notion of differential closure for characteristic p; i.e., of whether prime differentially closed extensions of models of $\tilde{T}_p$ exist. This is answered affirmatively in a forthcoming paper, where we use an algebraic characterization of certain extensions of models of $\tilde{T}_p$ to show prime model extensions exist, and also to give simple axioms for $\tilde{T}_p^*$ (and hence for T_p^*), which we were unable to do in §3 of the present paper.

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