A CHARACTERIZATION OF FOUR-DIMENSIONAL EINSTEIN MANIFOLDS

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ABSTRACT. It is shown that for any P on a four-dimensional Einstein manifold M , there exists an orthonormal basis $\{Y_i\}$ in $T_p(M)$ such that all the sectional curvatures $K(Y_i, Y_j)$ are equal.

1. Let M be an n -dimensional Riemannian manifold with inner product g . A covariant curvature tensor with respect to g at every point $P \in M$ is [1] a multilinear map $R: T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \rightarrow R$, or

$$(1) \quad R(X_i, X_j, X_k, X_l) = R_{ijkl}$$

for an orthonormal basis $\{x_i\}$ in the tangent space $T_p(M)$, where the curvature tensor satisfies the usual identities

$$(2) \quad R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij},$$

$$(3) \quad R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

We define the sectional curvature K as follows: If $\sigma = \{X, Y\}$ is a 2-plane $\subseteq T_p(M)$, then

$$(4) \quad K(X, Y) = \frac{R(X, Y, Y, X)}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2},$$

where $\langle X, Y \rangle = g(X, Y)$ is the inner product of X, Y . In case X_i, X_j are orthonormal,

$$(5) \quad K(X_i, X_j) = R(X_i, X_j, X_j, X_i) = -R_{ijij}.$$

A Ricci tensor is a bilinear map defined by

$$(6) \quad \text{Ric}(Y, Z) = \sum_{i=1}^n R(X_i, Y, Z, X_i),$$

$\{X_i\}$ being an orthonormal basis of $T_p(M)$.

An Einstein manifold M is a Riemannian manifold with metric g satisfying

$$(7) \quad \text{Ric}(Y, Z) = ag(Y, Z).$$

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where a is a constant. By taking an orthonormal basis $\{X_i\}$ in $T_p(M)$ at a point $p \in M$, we have $\text{Ric}(X_j, X_k) = ag(X_j, X_k) = a\delta_{jk}$ for an Einstein manifold, where δ_{jk} is the Kronecker delta.

2. Before discussing our theorem, we introduce some lemmas as follows; the first two are well known.

LEMMA 1. *The sectional curvature of a 2-plane σ is independent of the choice of basis vectors of σ .*

LEMMA 2 (SCHUR). *The sectional curvature is a constant K_0 if and only if*

$$(8) \quad R_{ijkl} = K_0(g_{il}g_{jk} - g_{ik}g_{jl}).$$

LEMMA 3 [1]. *Let $\{X_1, \dots, X_4\}$ be an orthonormal basis of the tangent space T_p of a four-dimensional Einstein manifold, and denote $K(X_i, X_j)$ by K_{ij} . Then*

$$(9) \quad K_{12} = K_{34}, \quad K_{13} = K_{24}, \quad K_{14} = K_{23}.$$

PROOF. From (1), (6), (7) it follows that $\text{Ric}(X_j, X_k) = \sum_{i=1}^4 R_{ijki} = a\delta_{jk}$, which becomes, for $j=k=1$, $R_{2112} + R_{3113} + R_{4114} = a$, or

$$(10) \quad K_{12} + K_{13} + K_{14} = a.$$

Similarly, we have

$$(11) \quad K_{21} + K_{23} + K_{24} = K_{31} + K_{32} + K_{34} = K_{41} + K_{42} + K_{43} = a.$$

Noting $K_{ij} = K_{ji}$ from the definition of the sectional curvature, and adding (10), (11) together, we obtain

$$(12) \quad K_{12} + K_{13} + K_{14} + K_{23} + K_{24} + K_{34} = 2a.$$

Again, from (10) and (11) it follows that

$$(13) \quad (K_{12} + K_{13} + K_{14}) + (K_{12} + K_{23} + K_{24}) = 2a,$$

which, together with (12), implies $K_{12} = K_{34}$.

Similar computations yield the other two equalities of (9).

LEMMA 4 [1]. *There exists an orthonormal basis $\{X_i\}$ of $T_p(M)$ of a four-dimensional Einstein manifold M such that all components, except $K(X_j, X_k)$, R_{1234} , R_{1324} , R_{1423} , of the curvature tensor are zero.*

PROOF. The possible nonzero components of R_{ijkl} ($1 \leq i, j, k, l \leq 4$) are K_{jk} , R_{ikjk} (j, k distinct and $\neq i$), and R_{ijkl} (with distinct i, j, k, l). We shall show that all R_{ikjk} are zero.

Let the 2-plane $\sigma(X_1, X_2) \subseteq T_p(M)$ be so chosen that $K(X_1, X_2)$ is the extremal value of the sectional curvatures at $P \in M$. This is possible since

orthonormal bases $\{X, Y\}$ form a closed set in $T_p(M)$ topologically and $K(X, Y)$ is a continuous function defined there.

Let the 2-plane $\zeta(X_3, X_4) \subseteq T_p(M)$ be the orthogonal complement of $\sigma(X_1, X_2)$; X_3 and X_4 can be arbitrarily chosen in this 2-plane. For any $Y \in \sigma(X_1, X_2)$, $Z \in \zeta(X_3, X_4)$, let X_1, X_3 be the vectors which give extremal values of $K(Y, Z)$. Then X_2, X_4 are determined naturally. Since $K(X_1, X_2)$ is extremal,

$$(14) \quad \begin{aligned} &K(X_1, X_2 \cos \theta + X_3 \sin \theta), & K(X_1, X_2 \cos \theta + X_4 \sin \theta), \\ &K(X_2, X_1 \cos \theta + X_3 \sin \theta), & K(X_2, X_1 \cos \theta + X_4 \sin \theta) \end{aligned}$$

are stationary at $\theta=0$, e.g., by setting

$$f(\theta) = K(X_1, X_2 \cos \theta + X_3 \sin \theta) = K_{12} \cos^2 \theta - R_{1213} \sin 2\theta + K_{13} \sin^2 \theta,$$

we have $f'(\theta) = -K_{12} \sin 2\theta - 2R_{1213} \cos 2\theta + K_{13} \sin 2\theta$, which is zero at $\theta=0$, so that $f'(0) = -2R_{1213} = 0$. Thus for the four given functions of θ given in (14), we obtain

$$(15) \quad R_{1213} + R_{1214} = R_{1223} = R_{1224} = 0.$$

Similarly, since $K(X_1, X_3)$ is extremal,

$$(16) \quad K(X_1, X_3 \cos \theta + X_4 \sin \theta), \quad K(X_3, X_1 \cos \theta + X_2 \sin \theta)$$

take stationary values at $\theta=0$, and therefore

$$(17) \quad R_{1314} = R_{1323} = 0.$$

On the other hand, $\text{Ric}(X_j, X_k) = \sum_{i=1}^4 R_{ijk l} = a \delta_{jk}$, which becomes for $j=2$ and $k=3$, $R_{1213} + R_{2434} = 0$, so that

$$(18) \quad R_{2434} = 0,$$

since $R_{1213} = 0$. By similar computations we obtain

$$(19) \quad R_{2334} = R_{1434} = R_{1334} = R_{2324} = R_{1424} = 0.$$

By (2), (15), (17), (18) and (19), the possible nonzero R_{ijkl} are thus K_{jk} ($1 \leq j < k \leq 4$), R_{1234} , R_{1324} , and R_{1423} for such basis $\{X_i\}$.

3. THEOREM. *There exists an orthonormal basis $\{Y_i\}$ of $T_p(M)$ of a four-dimensional Einstein manifold M such that all sectional curvatures $K(Y_j, Y_k)$ are equal.*

PROOF. It should be remarked that if K is constant, every orthonormal basis $\{Y_i\}$ of $T_p(M)$ is the required one.

For the general case, we choose $\{X_i\}$ to be an orthonormal basis as stated in Lemma 4, so that

$$(20) \quad R_{ikjk} = 0 \quad (i \neq j).$$

In view of Lemma 3, we need only to show that $K(Y_1, Y_2) = K(Y_2, Y_3) = K(Y_3, Y_1)$ for the required basis $\{Y_i\}$. To this end we let $K_{12} \geq K_{23} \geq K_{31}$. Consider an orthogonal transformation of $O(4)$, which keeps X_4 fixed and rotates about the triple $\{X_1, X_2, X_3\}$. Let this transformation be of the form $Y_i = TX_i$, $i=1, \dots, 4$. Since $Y_4 = X_4$, $T = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ with $Y_i = AX_i$, $i=1, 2, 3$, and $A = (\alpha_{jk})$ is an orthogonal matrix. Let $\text{Adj } A = (\beta_{jk})$, $K_{12} = K_{23} + \varepsilon_1$, $\varepsilon_1 \geq 0$; $K_{13} = K_{23} - \varepsilon_2$, $\varepsilon_2 \geq 0$; and $R_{jk} = K(Y_j, Y_k)$. Then by (20),

$$\begin{aligned} R_{12} &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})^2 K_{12} + (\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22})^2 K_{23} \\ &\quad + (\alpha_{13}\alpha_{21} - \alpha_{11}\alpha_{23})^2 K_{13} \\ (21) \quad &= \beta_{33}^2 K_{12} + \beta_{13}^2 K_{23} + \beta_{23}^2 K_{13}, \quad \left(\text{since } \sum_{j=1}^3 \beta_{j3}^2 = 1 \right) \\ &= K_{23} + \beta_{33}^2 \varepsilon_1 - \beta_{23}^2 \varepsilon_2. \end{aligned}$$

Similarly,

$$(22) \quad R_{23} = K_{23} + \beta_{31}^2 \varepsilon_1 - \beta_{21}^2 \varepsilon_2, \quad R_{31} = K_{23} + \beta_{32}^2 \varepsilon_1 - \beta_{22}^2 \varepsilon_2.$$

Thus $R_{12} = R_{23} = R_{31}$ provided that

$$\begin{aligned} (23) \quad \beta_{3k}^2 \varepsilon_1 - \beta_{2k}^2 \varepsilon_2 &= \frac{1}{3} [\varepsilon_1 (\beta_{31}^2 + \beta_{32}^2 + \beta_{33}^2) - \varepsilon_2 (\beta_{23}^2 + \beta_{21}^2 + \beta_{22}^2)] \\ &= \frac{1}{3} (\varepsilon_1 - \varepsilon_2), \quad k = 1, 2, 3. \end{aligned}$$

Among the three equations of (23), only two of them are linearly independent due to the facts that $\sum_{k=1}^3 \beta_{jk}^2 = 1$, $j=2, 3$. Since the orthogonality of the matrix (β_{jk}) is given by

$$(24) \quad \sum_{k=1}^3 \beta_{jk} \beta_{lk} = \delta_{jl}, \quad j, l = 1, 2, 3,$$

(β_{jk}) can be found by the two independent equations of (23) and the six equations of (24) indicated as follows:

$(\beta_{11}, \beta_{21}, \beta_{31})$ can be found arbitrarily by the equations

$$(25) \quad \varepsilon_1 \beta_{31}^2 - \varepsilon_2 \beta_{21}^2 = \frac{1}{3} (\varepsilon_1 - \varepsilon_2), \quad \beta_{11}^2 + \beta_{21}^2 + \beta_{31}^2 = 1,$$

and for one such solution, $(\beta_{12}, \beta_{22}, \beta_{32})$ is determined by

$$\begin{aligned} &\beta_{11}\beta_{12} + \beta_{21}\beta_{22} + \beta_{31}\beta_{32} = 0, \\ (26) \quad &\varepsilon_1 \beta_{32}^2 - \varepsilon_2 \beta_{22}^2 = \frac{1}{3} (\varepsilon_1 - \varepsilon_2), \\ &\beta_{12}^2 + \beta_{22}^2 + \beta_{32}^2 = 1. \end{aligned}$$

Then $(\beta_{13}, \beta_{23}, \beta_{33})$ follows from

$$(27) \quad \begin{aligned} \beta_{11}\beta_{13} + \beta_{21}\beta_{23} + \beta_{31}\beta_{33} &= 0, \\ \beta_{12}\beta_{13} + \beta_{22}\beta_{23} + \beta_{32}\beta_{33} &= 0, \\ \beta_{13}^2 + \beta_{23}^2 + \beta_{33}^2 &= 1. \end{aligned}$$

For such orthogonal matrix $\text{Adj } A = (\beta_{jk})$, $A = (\alpha_{jk}) = (\text{Adj } A)^{-1}|A| = (\text{Adj } A)^t = (\beta_{kj})$ is thus obtained; e.g., for $\varepsilon_2 = 0$ (i.e., $K_{12} > K_{23} = K_{31}$), or $\varepsilon_1 = \varepsilon_2$ (i.e., $K_{23} = \frac{1}{2}(K_{12} + K_{31})$), we have respectively

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

or

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

by the above method.

REMARK. By Lemma 1, the orthonormal basis $\{Y_i\}$ may be replaced by an orthogonal basis $\{\tilde{Y}_i\}$, where $\tilde{Y}_i$ are parallel to Y_i .

REFERENCE

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